THE KAC-WANG-YAN ALGEBRA WITH NEGATIVE INTEGRAL CENTRAL CHARGE

ANDREW R. LINSHAW

ABSTRACT. The Lie algebra $\mathcal{D}$ of regular differential operators on the circle has a universal central extension $\hat{\mathcal{D}}$. The invariant subalgebra $\hat{\mathcal{D}}^+$ under an involution preserving the principal gradation was introduced by Kac-Wang-Yan. The vacuum $\hat{\mathcal{D}}^+$-module with central charge $c \in \mathbb{C}$, and its irreducible quotient $\mathcal{V}_c$, possess vertex algebra structures. We prove that for every integer $n > 0$, $\mathcal{V}_{-n}$ is a $\mathcal{W}$-algebra of type $\mathcal{W}(2, 4, 6, \dots, 2n^2+4n)$. This is a formal consequence of Weyl's first and second fundamental theorems of invariant theory for the symplectic group Sp_{2n}. Our result implies that invariant subalgebras of the $\beta\gamma$-system of rank n under arbitrary reductive group actions are strongly finitely generated.

CONTENTS

1. INTRODUCTION

Let $\mathcal{D}$ denote the Lie algebra of regular differential operators on the circle. It has a universal central extension $\hat{\mathcal{D}} = \mathcal{D} \oplus \mathbb{C}\kappa$ which was introduced by Kac-Peterson in [KP]. Although $\hat{\mathcal{D}}$ admits a principal $\mathbb{Z}$-gradation and triangular decomposition, its representation theory is nontrivial because the graded pieces are all infinite-dimensional. The important problem of constructing and classifying the *quasi-finite* irreducible, highest-weight representations (i.e., those with finite-dimensional graded pieces) was solved by Kac-Radul in

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[KRI]. In [FKRW], the representation theory of $\hat{\mathcal{D}}$ was developed by Frenkel-Kac-Radul-Wang from the point of view of vertex algebras. For each $c \in \mathbb{C}$, $\hat{\mathcal{D}}$ admits a module $\mathcal{M}_c$ called the *vacuum module*, which is a vertex algebra freely generated by fields J^l of weight $l+1$, for $l \geq 0$. The highest-weight representations of $\hat{\mathcal{D}}$ are in one-to-one correspondence with the highest-weight representations of $\mathcal{M}_c$. The irreducible quotient of $\mathcal{M}_c$ by its maximal graded, proper $\hat{\mathcal{D}}$-submodule is a simple vertex algebra, and is often denoted by $\mathcal{W}_{1+\infty,c}$. These algebras have been studied extensively in both the physics and mathematics literature and they play an important role the theory of integrable systems. For generic values of c, $\mathcal{M}_c$ is irreducible, so $\mathcal{W}_{1+\infty,c} \cong \mathcal{M}_c$, but when c is an integer, $\mathcal{M}_c$ is reducible, and the structure and representation theory of $\mathcal{W}_{1+\infty,c}$ are nontrivial.

In [KWY], it was shown that there are two anti-involutions $\sigma_\pm$ of $\hat{\mathcal{D}}$ (up to conjugation) which preserve the principal $\mathbb{Z}$-gradation. The invariant Lie subalgebras under these involutions are denoted by $\hat{\mathcal{D}}^\pm$, and the authors regard $\hat{\mathcal{D}}^+$ as the more fundamental of these two subalgebras. For each $c \in \mathbb{C}$, $\hat{\mathcal{D}}^+$ admits a vacuum module $\mathcal{M}_c^+$ which is a vertex subalgebra of $\mathcal{M}_c$, and is freely generated by fields W^{2k+1} of weight $2k+2$, for $k \geq 0$. The unique irreducible quotient of $\mathcal{M}_c^+$ is denoted by $\mathcal{V}_c$ in [KWY], and we shall refer to $\mathcal{V}_c$ as the *Kac-Wang-Yan algebra* in this paper. Let π_c denote the projection $\mathcal{M}_c^+ \to \mathcal{V}_c$, whose kernel $\mathcal{I}_c$ is the maximal proper graded $\hat{\mathcal{D}}^+$-submodule of $\mathcal{M}_c^+$, and let $w^{2k+1} = \pi_c(W^{2k+1})$. Generically $\mathcal{M}_c^+$ is irreducible, but when c is an integer or a half-integer, $\mathcal{M}_c^+$ is reducible, and the structure and representation theory of $\mathcal{V}_c$ are nontrivial.

When c is a positive integer n or a positive half-integer $n+1/2$ the structures of both $\mathcal{V}_n$ and $\mathcal{V}_{n+1/2}$ are well understood. By Theorem 14.2 of [KWY], the $\mathcal{W}$-algebra $\mathcal{W}\mathcal{D}_n$ of central charge n decomposes as the direct sum of $\mathcal{V}_n$ and an irreducible, highest-weight $\mathcal{V}_n$-module. A consequence is that $\mathcal{V}_n$ has a minimal strong generating set

$$\{w^1, w^3, w^5, \dots, w^{2n-3}\},$$

and is therefore a $\mathcal{W}$-algebra of type $\mathcal{W}(2,4,6,\dots,2n-2)$. Similarly, by Theorem 14.4 of [KWY] the $\mathcal{W}$-superalgebra $\mathcal{W}\mathcal{B}(0,n)$ of central charge $n-1/2$ is the direct sum of $\mathcal{V}_{n+1/2}$ and an irreducible, highest-weight $\mathcal{V}_{n+1/2}$-module. This implies that $\mathcal{V}_{n+1/2}$ is a $\mathcal{W}$-algebra of type $\mathcal{W}(2,4,6,\dots,2n)$.

In the case where n is a negative integer, it is an open problem to determine the structure of $\mathcal{V}_{-n}$ and $\mathcal{V}_{-n+1/2}$, and to find a minimal strong generating set; see Remark 14.6 of [KWY]. In this paper we focus on $\mathcal{V}_{-n}$, and our starting point is a remarkable free field realization of $\mathcal{V}_{-n}$ as the Sp_{2n}-invariant subalgebra of the vertex algebra $\mathcal{F}^{\otimes -n}$, which is just the $\beta\gamma$-system of rank n [KWY]. In order to be consistent with the notation in [LI] and [LII], we shall denote $\mathcal{F}^{\otimes -n}$ by $\mathcal{S}(V)$, where V is the vector space $\mathbb{C}^n$. The isomorphism $\mathcal{V}_{-n} \cong \mathcal{S}(V)^{Sp_{2n}}$ indicates that the structure of $\mathcal{V}_{-n}$ is deeply connected to *classical invariant theory*. The action of Sp_{2n} on $\mathcal{S}(V)$ is analogous to the action of Sp_{2n} on the Weyl algebra $\mathcal{D}(V)$, and Sp_{2n} is the *full* automorphism group of $\mathcal{S}(V)$. Moreover, $\mathcal{S}(V)$ admits an Sp_{2n}-invariant filtration such that the associated graded object $\text{gr}(\mathcal{S}(V))$ is isomorphic to $\text{Sym} \bigoplus_{k\geq 0} U_k$ as a commutative ring, where each U_k is a copy of the standard Sp_{2n}-representation $\mathbb{C}^{2n}$. As a vector space, $\mathcal{V}_{-n}$ is isomorphic to $R = (\text{Sym} \bigoplus_{k\geq 0} U_k)^{Sp_{2n}}$, and we have isomorphisms of commutative rings

$$\text{gr}(\mathcal{V}_{-n}) \cong \text{gr}(\mathcal{S}(V)^{Sp_{2n}}) \cong \text{gr}(\mathcal{S}(V))^{Sp_{2n}} \cong R.$$

In this sense, we regard $\mathcal{V}_{-n}$ as a *deformation* of the classical invariant ring R.

By Weyl's first and second fundamental theorems of invariant theory for the standard representation of Sp_{2n}, R is generated by quadratics $\{q_{a,b}|\ 0 \leq a < b\}$, and the ideal of relations among the $q_{a,b}$'s is generated by Pfaffians of degree $n+1$. We obtain a corresponding strong generating set $\{\omega_{a,b}|\ 0 \leq a < b\}$ for $\mathcal{V}_{-n}$, as well as a generating set for the ideal of relations among the $\omega_{a,b}$'s, which correspond to the Pfaffians with suitable quantum corrections. This generating set is related to $\{\partial^k w^{2m+1}|k, m \geq 0\}$ by a linear change of variables. The relation of minimal weight among the generators occurs at weight $2(n+1)^2$, and corresponds to a singular vector $P_0 \in \mathcal{I}_{-n} \subset \mathcal{M}_{-n}^+$. In fact, we shall see that P_0 generates $\mathcal{I}_{-n}$ as a vertex algebra ideal. This is analogous to Theorem 4.4 of [LI], which states that the maximal proper graded $\hat{\mathcal{D}}$-submodule of $\mathcal{M}_{-n}$ is generated as a vertex algebra ideal by a singular vector of weight $(n+1)^2$.

The technical heart of this paper is a detailed analysis of the quantum corrections of the above classical relations. We will prove that the relation in $\mathcal{V}_{-n}$ of minimal weight is of the form

$$w^{2n^2+4n+1} = Q(w^1, w^3, w^5, \dots, w^{2n^2+4n-1}), \tag{1.1}$$

where Q is a normally ordered polynomial in $w^1, w^3, w^5, \dots, w^{2n^2+4n-1}$ and their derivatives. We call (1.1) a *decoupling relation*, and by applying the operator $w^3 \circ_1$ repeatedly, it is easy to construct higher decoupling relations

$$w^{2m+1} = Q_m(w^1, w^3, w^5, \dots, w^{2n^2+4n-1})$$

for all $m > n^2 + 2n$. This shows that $\{w^1, w^3, w^5, \dots, w^{2n^2+4n-1}\}$ is a minimal strong generating set for $\mathcal{V}_{-n}$, and in particular $\mathcal{V}_{-n}$ is a $\mathcal{W}$-algebra of type $\mathcal{W}(2, 4, 6, \dots, 2n^2 + 4n)$. This generalizes a theorem of de Boer-Feher-Honecker in the case $n = 1$ (see Section 1 of [BFH]), and is analogous to Theorem 4.16 of [LI], which states that $\mathcal{W}_{1+\infty,-n}$ is a $\mathcal{W}$-algebra of type $\mathcal{W}(1, 2, 3, \dots, n^2 + 2n)$.

Given a simple, finite-dimensional Lie algebra $\mathfrak{g}$, let $V_k(\mathfrak{g})$ denote the corresponding irreducible affine vertex algebra at level k. By Theorem 14.5 of [KWY], $\mathcal{V}_{-n}$ is isomorphic to the Sp_{2n}-invariant subalgebra $V_{-1/2}(\mathfrak{sp}_{2n})^{Sp_{2n}}$, which was denoted by $\mathcal{W}(C_n^{(1)}/C_n, -1)$ in [KWY]. It is known (see Equation 2.4 of [BFH]) that the commutant of the diagonal $V_{m-1/2}(\mathfrak{sp}_{2n})$ inside $V_m(\mathfrak{sp}_{2n}) \otimes V_{-1/2}(\mathfrak{sp}_{2n})$, is a deformation of $V_{-1/2}(\mathfrak{sp}_{2n})^{Sp_{2n}}$, and

$$V_{-1/2}(\mathfrak{sp}_{2n})^{Sp_{2n}} \cong \lim_{m\to\infty} \text{Com}(V_{m-1/2}(\mathfrak{sp}_{2n}), V_m(\mathfrak{sp}_{2n}) \otimes V_{-1/2}(\mathfrak{sp}_{2n})).$$

This commutant was conjectured by Blumenhagen-Eholzer-Honecker-Hornfeck-Hubel (see Table 7 of [B-H]) to be a deformable $\mathcal{W}$-algebra of type $\mathcal{W}(2, 4, 6, \dots, 2n^2 + 4n)$, so our result was anticipated in the physics literature as early as 1995[1].

The representation theory of $\mathcal{V}_{-n}$ is governed by its Zhu algebra, which is a commutative algebra on generators $a^1, a^3, a^5, \dots, a^{2n^2+4n-1}$ corresponding to the generators of $\mathcal{V}_{-n}$. The irreducible, admissible $\mathbb{Z}_{\geq 0}$-graded $\mathcal{V}_{-n}$-modules are therefore all highest-weight modules, and are parametrized by the points in the variety Spec$(A(\mathcal{V}_{-n}))$, which is a subvariety of $\mathbb{C}^{n^2+2n}$. By Theorem 13.2 of [KWY], Sp_{2n} and $\mathcal{V}_{-n}$ form a dual reductive pair in the sense that the $\beta\gamma$-system $\mathcal{S}(V)$ of rank n admits a decomposition

$$\mathcal{S}(V) \cong \bigoplus_{\nu \in H} L(\nu) \otimes M^\nu, \tag{1.2}$$

[1] I thank T. Creutzig for pointing this out to me.

where H indexes the irreducible, finite-dimensional representations $L(\nu)$ of Sp_{2n}, and the M^{ν}'s are inequivalent, irreducible, highest-weight $\mathcal{V}_{-n}$-modules. The modules M^{ν} appearing in this decomposition have an integrality property, and they correspond to certain rational points on $\text{Spec}(A(\mathcal{V}_{-n}))$.

Our result on the structure of $\mathcal{V}_{-n}$ is the missing ingredient in our program of studying invariant subalgebras of the $\beta\gamma$-system under arbitrary reductive group actions [LII]. Recall that a vertex algebra $\mathcal{A}$ is called *strongly finitely generated* if there exists a finite set of generators such that the collection of iterated Wick products of the generators and their derivatives spans $\mathcal{A}$. This property has many important consequences, and in particular implies that the Zhu algebra of $\mathcal{A}$ is finitely generated. In recent work [LII][LIII][LIV], we have investigated the strong finite generation of invariant vertex algebras $\mathcal{A}^G$, where $\mathcal{A}$ is a free field or affine vertex algebra and G is a reductive group of automorphisms of $\mathcal{A}$. This is an analogue of Hilbert's theorem on the finite generation of classical invariant rings. It is a subtle and essentially "quantum" phenomenon that is generally destroyed by passing to the classical limit before taking invariants. Often, $\mathcal{A}$ admits a G-invariant filtration for which $\text{gr}(\mathcal{A})$ is a commutative algebra with a derivation, and the classical limit $\text{gr}(\mathcal{A}^G)$ is isomorphic to $(\text{gr}(\mathcal{A}))^G$ as a commutative algebra. Unlike $\mathcal{A}^G$, $\text{gr}(\mathcal{A}^G)$ is generally not finitely generated as a vertex algebra, and a presentation will require both infinitely many generators and infinitely many relations.

Isolated examples of this phenomenon have been known for many years (see for example [BFH][EFH][DN][FKRW][KWY]), although the first general results of this kind were obtained in [LII], in the case where $\mathcal{A}$ is the $\beta\gamma$-system $\mathcal{S}(V)$ of rank n, and G is a subgroup of GL_n. By a theorem of Kac-Radul [KRII] $\mathcal{S}(V)^{GL_n} \cong \mathcal{W}_{1+\infty,-n}$, and for any reductive $G \subset GL_n$, the structure of $\mathcal{S}(V)^G$ is governed by $\mathcal{W}_{1+\infty,-n}$ in the sense that $\mathcal{S}(V)^G$ decomposes as a direct sum of irreducible, highest-weight $\mathcal{W}_{1+\infty,-n}$-modules. Using this decomposition together the description of $\mathcal{W}_{1+\infty,-n}$ as a $\mathcal{W}$-algebra of type $\mathcal{W}(1,2,3,\dots,n^2+2n)$, we showed that $\mathcal{S}(V)^G$ is strongly finitely generated.

For an *arbitrary* reductive group G of automorphisms of $\mathcal{S}(V)$, the structure of $\mathcal{S}(V)^G$ is governed by $\mathcal{V}_{-n}$ rather than $\mathcal{W}_{1+\infty,-n}$, since $G \subset Sp_{2n}$. It is easy to show using (1.2) that $\mathcal{S}(V)^G$ decomposes as a direct sum of irreducible, highest-weight $\mathcal{V}_{-n}$-modules. Using our result that $\mathcal{V}_{-n}$ is a $\mathcal{W}$-algebra of type $\mathcal{W}(2,4,6,\dots,2n^2+4n)$, we will show that $\mathcal{S}(V)^G$ is strongly finitely generated. Our proof is essentially constructive, and it completes the study initiated in [LII] of the invariant subalgebras of the $\beta\gamma$-system.

2. Vertex algebras

In this section, we define vertex algebras, which have been discussed from various different points of view in the literature [B][FBZ][FHL][FLM][K][LiI][LZ]. We will follow the formalism developed in [LZ] and partly in [LiI]. Let $V = V_0 \oplus V_1$ be a super vector space over $\mathbb{C}$, and let z, w be formal variables. By $\text{QO}(V)$, we mean the space of all linear maps

$$V \to V((z)) := \{\sum_{n\in\mathbb{Z}} v(n)z^{-n-1} | v(n) \in V,\ v(n) = 0 \text{ for } n >> 0\}.$$

Each element $a \in \text{QO}(V)$ can be uniquely represented as a power series

$$a = a(z) := \sum_{n\in\mathbb{Z}} a(n)z^{-n-1} \in \text{End}(V)[[z, z^{-1}]].$$

We refer to $a(n)$ as the nth Fourier mode of $a(z)$. Each $a \in \mathrm{QO}(V)$ is of the shape $a = a_0 + a_1$ where $a_i : V_j \to V_{i+j}((z))$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$, and we write $|a_i| = i$.

On $\mathrm{QO}(V)$ there is a set of nonassociative bilinear operations $\circ_n$, indexed by $n \in \mathbb{Z}$, which we call the nth circle products. For homogeneous $a, b \in \mathrm{QO}(V)$, they are defined by

$$a(w) \circ_n b(w) = \mathrm{Res}_z a(z)b(w)\ \iota_{|z|>|w|}(z-w)^n - (-1)^{|a||b|}\mathrm{Res}_z b(w)a(z)\ \iota_{|w|>|z|}(z-w)^n.$$

Here $\iota_{|z|>|w|} f(z,w) \in \mathbb{C}[[z, z^{-1}, w, w^{-1}]]$ denotes the power series expansion of a rational function f in the region $|z| > |w|$. We usually omit the symbol $\iota_{|z|>|w|}$ and just write $(z-w)^{-1}$ to mean the expansion in the region $|z| > |w|$, and write $-(w-z)^{-1}$ to mean the expansion in $|w| > |z|$. It is easy to check that $a(w) \circ_n b(w)$ above is a well-defined element of $\mathrm{QO}(V)$.

The nonnegative circle products are connected through the *operator product expansion* (OPE) formula. For $a, b \in \mathrm{QO}(V)$, we have

$$a(z)b(w) = \sum_{n \geq 0} a(w) \circ_n b(w)\ (z-w)^{-n-1} + :a(z)b(w): , \tag{2.1}$$

which is often written as $a(z)b(w) \sim \sum_{n\geq 0} a(w) \circ_n b(w)\ (z-w)^{-n-1}$, where $\sim$ means equal modulo the term

$$:a(z)b(w): = a(z)_- b(w)\ +\ (-1)^{|a||b|} b(w) a(z)_+.$$

Here $a(z)_- = \sum_{n<0} a(n) z^{-n-1}$ and $a(z)_+ = \sum_{n\geq 0} a(n) z^{-n-1}$. Note that $:a(w)b(w):$ is a well-defined element of $\mathrm{QO}(V)$. It is called the *Wick product* of a and b, and it coincides with $a \circ_{-1} b$. The other negative circle products are related to this by

$$n!\ a(z) \circ_{-n-1} b(z) = :(\partial^n a(z)) b(z): ,$$

where ∂ denotes the formal differentiation operator $\frac{d}{dz}$. For $a_1(z), \dots, a_k(z) \in \mathrm{QO}(V)$, the k-fold iterated Wick product is defined to be

$$:a_1(z)a_2(z)\cdots a_k(z): = :a_1(z)b(z): , \tag{2.2}$$

where $b(z) = :a_2(z)\cdots a_k(z):$. We often omit the formal variable z when no confusion can arise.

The set $\mathrm{QO}(V)$ is a nonassociative algebra with the operations $\circ_n$, which satisfy $1 \circ_n a = \delta_{n,-1} a$ for all n, and $a \circ_n 1 = \delta_{n,-1} a$ for $n \geq -1$. In particular, 1 behaves as a unit with respect to $\circ_{-1}$. A linear subspace $\mathcal{A} \subset \mathrm{QO}(V)$ containing 1 which is closed under the circle products will be called a *quantum operator algebra* (QOA). Note that $\mathcal{A}$ is closed under ∂ since $\partial a = a \circ_{-2} 1$. Many formal algebraic notions are immediately clear: a homomorphism is just a linear map that sends 1 to 1 and preserves all circle products; a module over $\mathcal{A}$ is a vector space M equipped with a homomorphism $\mathcal{A} \to \mathrm{QO}(M)$, etc. A subset $S = \{a_i |\ i \in I\}$ of $\mathcal{A}$ is said to generate $\mathcal{A}$ if every element $a \in \mathcal{A}$ can be written as a linear combination of nonassociative words in the letters a_i, $\circ_n$, for $i \in I$ and $n \in \mathbb{Z}$. We say that S *strongly generates* $\mathcal{A}$ if every $a \in \mathcal{A}$ can be written as a linear combination of words in the letters a_i, $\circ_n$ for $n < 0$. Equivalently, $\mathcal{A}$ is spanned by the collection $\{:\partial^{k_1} a_{i_1}(z) \cdots \partial^{k_m} a_{i_m}(z):\ |\ i_1, \dots, i_m \in I,\ k_1, \dots, k_m \geq 0\}$.

We say that $a, b \in \mathrm{QO}(V)$ *quantum commute* if $(z-w)^N [a(z), b(w)] = 0$ for some $N \geq 0$. Here $[,]$ denotes the super bracket. This condition implies that $a \circ_n b = 0$ for $n \geq N$, so (2.1) becomes a finite sum. A *commutative quantum operator algebra* (CQOA) is a QOA whose elements pairwise quantum commute. Finally, the notion of a CQOA is equivalent

to the notion of a vertex algebra. Every CQOA $\mathcal{A}$ is itself a faithful $\mathcal{A}$-module, called the *left regular module*. Define

$$\rho : \mathcal{A} \to \mathrm{QO}(\mathcal{A}), \quad a \mapsto \hat{a}, \quad \hat{a}(\zeta)b = \sum_{n\in\mathbb{Z}} (a \circ_n b)\, \zeta^{-n-1}.$$

Then ρ is an injective QOA homomorphism, and the quadruple of structures $(\mathcal{A}, \rho, 1, \partial)$ is a vertex algebra in the sense of [FLM]. Conversely, if $(V, Y, \mathbf{1}, D)$ is a vertex algebra, the collection $Y(V) \subset \mathrm{QO}(V)$ is a CQOA. *We will refer to a CQOA simply as a vertex algebra throughout the rest of this paper.*

3. Category $\mathcal{R}$

Let $\mathcal{R}$ be the category of vertex algebras $\mathcal{A}$ equipped with a $\mathbb{Z}_{\geq 0}$-filtration

$$\mathcal{A}_{(0)} \subset \mathcal{A}_{(1)} \subset \mathcal{A}_{(2)} \subset \cdots, \qquad \mathcal{A} = \bigcup_{k\geq 0} \mathcal{A}_{(k)} \tag{3.1}$$

such that $\mathcal{A}_{(0)} = \mathbb{C}$, and for all $a \in \mathcal{A}_{(k)}$, $b \in \mathcal{A}_{(l)}$, we have

$$a \circ_n b \in \mathcal{A}_{(k+l)}, \quad \text{for } n < 0, \tag{3.2}$$

$$a \circ_n b \in \mathcal{A}_{(k+l-1)}, \quad \text{for } n \geq 0. \tag{3.3}$$

Elements $a(z) \in \mathcal{A}_{(d)} \setminus \mathcal{A}_{(d-1)}$ are said to have degree d.

Filtrations on vertex algebras satisfying (3.2)-(3.3) were introduced in [LiII], and are known as *good increasing filtrations*. Setting $\mathcal{A}_{(-1)} = \{0\}$, the associated graded object $\mathrm{gr}(\mathcal{A}) = \bigoplus_{k\geq 0} \mathcal{A}_{(k)}/\mathcal{A}_{(k-1)}$ is a $\mathbb{Z}_{\geq 0}$-graded associative, (super)commutative algebra with a unit 1 under a product induced by the Wick product on $\mathcal{A}$. In general, there is no natural linear map $\mathcal{A} \to \mathrm{gr}(\mathcal{A})$, but for each $r \geq 1$ we have the projection

$$\phi_r : \mathcal{A}_{(r)} \to \mathcal{A}_{(r)}/\mathcal{A}_{(r-1)} \subset \mathrm{gr}(\mathcal{A}). \tag{3.4}$$

Moreover, $\mathrm{gr}(\mathcal{A})$ has a derivation ∂ of degree zero (induced by the operator $\partial = \frac{d}{dz}$ on $\mathcal{A}$), and for each $a \in \mathcal{A}_{(d)}$ and $n \geq 0$, the operator $a\circ_n$ on $\mathcal{A}$ induces a derivation of degree $d - k$ on $\mathrm{gr}(\mathcal{A})$, which we denote by $a(n)$. Here

$$k = \sup\{j \geq 1 |\ \mathcal{A}_{(r)} \circ_n \mathcal{A}_{(s)} \subset \mathcal{A}_{(r+s-j)},\ \forall r, s, n \geq 0\},$$

as in [LL]. Finally, these derivations give $\mathrm{gr}(\mathcal{A})$ the structure of a vertex Poisson algebra.

The assignment $\mathcal{A} \mapsto \mathrm{gr}(\mathcal{A})$ is a functor from $\mathcal{R}$ to the category of $\mathbb{Z}_{\geq 0}$-graded (super)commutative rings with a differential ∂ of degree 0, which we will call ∂-rings. A ∂-ring is the same thing as an *abelian* vertex algebra, that is, a vertex algebra $\mathcal{V}$ in which $[a(z), b(w)] = 0$ for all $a, b \in \mathcal{V}$. A ∂-ring A is said to be generated by a subset $\{a_i |\ i \in I\}$ if $\{\partial^k a_i |\ i \in I, k \geq 0\}$ generates A as a graded ring. The key feature of $\mathcal{R}$ is the following reconstruction property [LL]:

Lemma 3.1. *Let $\mathcal{A}$ be a vertex algebra in $\mathcal{R}$ and let $\{a_i |\ i \in I\}$ be a set of generators for gr$(\mathcal{A})$ as a ∂-ring, where a_i is homogeneous of degree d_i. If $a_i(z) \in \mathcal{A}_{(d_i)}$ are vertex operators such that $\phi_{d_i}(a_i(z)) = a_i$, then $\mathcal{A}$ is strongly generated as a vertex algebra by $\{a_i(z) |\ i \in I\}$.*

As shown in [LI], there is a similar reconstruction property for kernels of surjective morphisms in $\mathcal{R}$. Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism in $\mathcal{R}$ with kernel $\mathcal{J}$, such that f maps $\mathcal{A}_{(k)}$ onto $\mathcal{B}_{(k)}$ for all $k \geq 0$. The kernel J of the induced map $\mathrm{gr}(f) : \mathrm{gr}(\mathcal{A}) \to \mathrm{gr}(\mathcal{B})$ is

a homogeneous ∂-ideal (i.e., $\partial J \subset J$). A set $\{a_i|\ i \in I\}$ such that a_i is homogeneous of degree d_i is said to generate J as a ∂-ideal if $\{\partial^k a_i|\ i \in I,\ k \geq 0\}$ generates J as an ideal.

Lemma 3.2. *Let $\{a_i|i \in I\}$ be a generating set for J as a ∂-ideal, where a_i is homogeneous of degree d_i. Then there exist vertex operators $a_i(z) \in \mathcal{A}_{(d_i)}$ with $\phi_{d_i}(a_i(z)) = a_i$, such that $\{a_i(z)|\ i \in I\}$ generates $\mathcal{J}$ as a vertex algebra ideal.*

4. The vertex algebra $\mathcal{W}_{1+\infty,c}$

Let $\mathcal{D}$ be the Lie algebra of regular differential operators on the circle, with coordinate t. A standard basis for $\mathcal{D}$ is

$$J^l_k = -t^{l+k}(\partial_t)^l, \quad k \in \mathbb{Z}, \quad l \in \mathbb{Z}_{\geq 0},$$

where $\partial_t = \frac{d}{dt}$. $\mathcal{D}$ has a 2-cocycle given by

$$\Psi\bigg(f(t)(\partial_t)^m, g(t)(\partial_t)^n\bigg) = \frac{m!n!}{(m+n+1)!}\mathrm{Res}_{t=0} f^{(n+1)}(t)g^{(m)}(t)dt, \tag{4.1}$$

and a corresponding central extension $\hat{\mathcal{D}} = \mathcal{D} \oplus \mathbb{C}\kappa$, which was first studied by Kac-Peterson in [KP]. $\hat{\mathcal{D}}$ has a $\mathbb{Z}$-grading $\hat{\mathcal{D}} = \bigoplus_{j\in\mathbb{Z}} \hat{\mathcal{D}}_j$ by weight, given by

$$\mathrm{wt}(J^l_k) = k, \quad \mathrm{wt}(\kappa) = 0,$$

and a triangular decomposition $\hat{\mathcal{D}} = \hat{\mathcal{D}}_+ \oplus \hat{\mathcal{D}}_0 \oplus \hat{\mathcal{D}}_-$, where $\hat{\mathcal{D}}_\pm = \bigoplus_{j\in\pm\mathbb{N}} \hat{\mathcal{D}}_j$ and $\hat{\mathcal{D}}_0 = \mathcal{D}_0 \oplus \mathbb{C}\kappa$. For a fixed $c \in \mathbb{C}$ and $\lambda \in \mathcal{D}^*_0$, define the Verma module with central charge c over $\hat{\mathcal{D}}$ by

$$\mathcal{M}_c(\hat{\mathcal{D}}, \lambda) = U(\hat{\mathcal{D}}) \otimes_{U(\hat{\mathcal{D}}_0 \oplus \hat{\mathcal{D}}_+)} \mathbf{C}_\lambda,$$

where $\mathbf{C}_\lambda$ is the one-dimensional $\hat{\mathcal{D}}_0 \oplus \hat{\mathcal{D}}_+$-module on which κ acts by multiplication by c and $h \in \hat{\mathcal{D}}_0$ acts by multiplication by $\lambda(h)$, and $\hat{\mathcal{D}}_+$ acts by zero. Let $\mathcal{P}$ be the parabolic subalgebra of $\mathcal{D}$ consisting of differential operators which extend to all of $\mathbb{C}$, which has a basis $\{J^l_k|\ l \geq 0,\ l+k \geq 0\}$. The cocycle Ψ vanishes on $\mathcal{P}$, so $\mathcal{P}$ may be regarded as a subalgebra of $\hat{\mathcal{D}}$, and $\hat{\mathcal{D}}_0 \oplus \hat{\mathcal{D}}_+ \subset \hat{\mathcal{P}}$, where $\hat{\mathcal{P}} = \mathcal{P} \oplus \mathbb{C}\kappa$. The induced $\hat{\mathcal{D}}$-module

$$\mathcal{M}_c = U(\hat{\mathcal{D}}) \otimes_{U(\hat{\mathcal{P}})} \mathbf{C}_0$$

is then a quotient of $\mathcal{M}_c(\hat{\mathcal{D}}, 0)$, and is known as the *vacuum $\hat{\mathcal{D}}$-module of central charge c*. $\mathcal{M}_c$ has the structure of a vertex algebra which is generated by fields

$$J^l(z) = \sum_{k\in\mathbb{Z}} J^l_k z^{-k-l-1}, \quad l \geq 0$$

of weight $l+1$. The modes J^l_k represent $\hat{\mathcal{D}}$ on $\mathcal{M}_c$, and as in [LI], we rewrite these fields in the form

$$J^l(z) = \sum_{k\in\mathbb{Z}} J^l(k) z^{-k-1},$$

where $J^l(k) = J^l_{k-l}$. In fact, $\mathcal{M}_c$ is *freely* generated by $\{J^l(z)|\ l \geq 0\}$; the set of iterated Wick products

$$:\partial^{i_1} J^{l_1}(z) \cdots \partial^{i_r} J^{l_r}(z):,$$

such that $l_1 \leq \cdots \leq l_r$ and $i_a \leq i_b$ if $l_a = l_b$, forms a basis for $\mathcal{M}_c$.

A weight-homogeneous element $\omega \in \mathcal{M}_c$ is called a *singular vector* if $J^l \circ_k \omega = 0$ for all $k > l \geq 0$. The maximal proper $\hat{\mathcal{D}}$-submodule $\mathcal{I}_c$ is the vertex algebra ideal generated by all singular vectors $\omega \neq 1$, and the unique irreducible quotient $\mathcal{M}_c/\mathcal{I}_c$ is often denoted by $\mathcal{W}_{1+\infty,c}$ in the literature. For $c \notin \mathbb{Z}$, $\mathcal{M}_c$ is irreducible, so $\mathcal{W}_{1+\infty,c} = \mathcal{M}_c$, but for $n \in \mathbb{Z}$, $\mathcal{M}_n$ is reducible. For $n \geq 1$, $\mathcal{W}_{1+\infty,n}$ is known to be isomorphic to $\mathcal{W}(\mathfrak{gl}_n)$ with central charge n [FKRW]. In [W] it was shown that $\mathcal{W}_{1+\infty,-1}$ is isomorphic to $\mathcal{W}(\mathfrak{gl}_n)$ of central charge -2, and in particular has generators in weights $1, 2, 3$. This result was generalized in [LI]; for all $n \geq 1$, $\mathcal{W}_{1+\infty,-n}$ has a minimal strong generating set consisting of a field in each weight $1, 2, 3, \dots, n^2 + 2n$.

5. The vertex algebra $\mathcal{V}_c$

The Lie algebra $\mathcal{D}$ has an anti-involution $\sigma_{+,-1}$, and the fixed point subalgebra $\mathcal{D}^+$ has generators

$$W^m_k = -\frac{1}{2}\bigg(t^{k+m}(\partial_t)^m + (-1)^{m+1}(\partial_t)^m t^{k+m}\bigg), \quad k \in \mathbb{Z}, \quad m \in 1 + 2\mathbb{Z}_{\geq 0}.$$

Note that $\{W^1_k | k \in \mathbb{Z}\}$ spans a copy of the Virasoro Lie algebra. We use the same notation Ψ to denote the restriction of the cocycle Ψ to $\mathcal{D}^+$. Let $\hat{\mathcal{D}}^+$ be the corresponding central extension of $\mathcal{D}^+$, which is clearly a subalgebra of $\hat{\mathcal{D}}$. Let $\mathcal{P}^+ = \mathcal{P} \cap \mathcal{D}^+$, which is a subalgebra of $\hat{\mathcal{D}}^+$ since Ψ vanishes on $\mathcal{P}^+$. Clearly

$$\mathcal{P}^+ = \{W^m_k | m + k \geq 0,\ k \in \mathbb{Z},\ m \in 1 + 2\mathbb{Z}_{\geq 0}\}.$$

The induced $\hat{\mathcal{D}}^+$-module

$$\mathcal{M}^+_c = U(\hat{\mathcal{D}}^+) \otimes_{U(\hat{\mathcal{P}}^+)} \mathbf{C}_0$$

is known as the *vacuum $\hat{\mathcal{D}}^+$-module of central charge c*. $\mathcal{M}^+_c$ has the structure of a vertex algebra which is generated by fields

$$W^m(z) = \sum_{k \in \mathbb{Z}} W^m_k z^{-k-m-1}, \qquad m \in 1 + 2\mathbb{Z}_{\geq 0}$$

of weight $m + 1$. The modes W^m_k represent $\hat{\mathcal{D}}^+$ on $\mathcal{M}^+_c$, and as usual, we rewrite these fields in the form

$$W^m(z) = \sum_{k \in \mathbb{Z}} W^m(k) z^{-k-1},$$

where $W^m(k) = W^m_{k-m}$. Clearly $\mathcal{M}^+_c$ is a vertex subalgebra of $\mathcal{M}_c$, and $\mathcal{M}^+_c$ is freely generated by $\{W^m(z) |\ m \in 1 + 2\mathbb{Z}_{\geq 0}\}$; the set of iterated Wick products

$$:\partial^{i_1} W^{m_1}(z) \cdots \partial^{i_r} W^{m_r}(z):, \tag{5.1}$$

such that $m_1 \leq \cdots \leq m_r$ and $i_a \leq i_b$ if $m_a = m_b$, forms a basis for $\mathcal{M}^+_c$.

Define a filtration

$$(\mathcal{M}^+_c)_{(0)} \subset (\mathcal{M}^+_c)_{(1)} \subset \cdots$$

on $\mathcal{M}^+_c$ as follows: for $k \geq 0$, $(\mathcal{M}^+_c)_{(2k)}$ is the span of monomials of the form (5.1), for $r \leq k$, and $(\mathcal{M}^+_c)_{(2k+1)} = (\mathcal{M}^+_c)_{(2k)}$. In particular, each W^m and its derivatives has degree 2. Equipped with this filtration, $\mathcal{M}^+_c$ lies in the category $\mathcal{R}$, and $\text{gr}(\mathcal{M}^+_c)$ is the polynomial algebra $\mathbb{C}[\partial^k W^{2m+1} |\ k, m \geq 0]$. Each element $W^{2m+1}(k) \in \mathcal{P}^+$ for $k, m \geq 0$ gives rise to a derivation of degree zero on $\text{gr}(\mathcal{M}^+_c)$ coming from the vertex Poisson algebra structure, and this action of $\mathcal{P}^+$ on $\text{gr}(\mathcal{M}^+_c)$ is independent of c.

Lemma 5.1. *For each $c \in \mathbb{C}$, $\mathcal{M}_c^+$ is generated as a vertex algebra by W^3.*

Proof. First, $W^1 = \frac{1}{360} W^3 \circ_5 W^3$, so W^1 lies in the subalgebra $\langle W^3 \rangle$ generated by W^3. Next, an OPE calculation shows that for all $m > 0$,

$$W^3 \circ_1 W^{2m+1} \equiv -(2m+2) W^{2m+3}, \tag{5.2}$$

modulo a linear combination of terms of the form $\partial^{2k} W^{2m+3-2k}$ for $1 \leq k \leq m+1$. It follows by induction on m that each $W^{2m+1} \in \langle W^3 \rangle$. $\square$

In particular, $\mathcal{M}_c^+$ is a finitely generated vertex algebra. However, $\mathcal{M}_c^+$ is not *strongly* generated by any finite set of vertex operators. This follows from the fact that $\text{gr}(\mathcal{M}_c^+)$ is the polynomial algebra with generators $\partial^k W^{2m+1}$ for $k, m \geq 0$, which implies that there are no normally ordered polynomial relations in $\mathcal{M}_c^+$. A weight-homogeneous element $\omega \in \mathcal{M}_c^+$ is called a *singular vector* if $W^{2m+1} \circ_k \omega = 0$ for all $m \geq 0$ and $k > 2m+1$. The maximal proper $\hat{\mathcal{D}}^+$-submodule $\mathcal{I}_c$ is the vertex algebra ideal generated by all singular vectors $\omega \neq 1$, and the Kac-Wang-Yan algebra $\mathcal{V}_c$ is the unique irreducible quotient $\mathcal{M}_c^+/\mathcal{I}_c$. We denote the projection $\mathcal{M}_c^+ \to \mathcal{V}_c$ by π_c, and we use the notation

$$w^{2m+1} = \pi_c(W^{2m+1}), \qquad m \geq 0, \tag{5.3}$$

in order to distinguish between $W^{m+1} \in \mathcal{M}_c^+$ and its image in $\mathcal{V}_c$. Clearly $\mathcal{V}_c$ is generated as a vertex algebra by w^3, but there may now be normally ordered polynomial relations among $\{w^{2m+1} | m \geq 0\}$ and their derivatives.

Recall that $\mathcal{M}_c^+$ is reducible if and only if c is an integer or a half-integer. In the case where c is a positive integer n or a positive half-integer $n + 1/2$, it follows from Theorems 14.2 and 14.4 of [KWY] that $\mathcal{V}_n$ and $\mathcal{V}_{n+1/2}$ are $\mathcal{W}$-algebras of types $\mathcal{W}(2, 4, 6, \dots, 2n-2)$ and $\mathcal{W}(2, 4, 6, \dots, 2n)$, respectively. However, finding a minimal strong generating set for $\mathcal{V}_c$ when c is a negative integer or half-integer is an open problem; see Remark 14.6 of [KWY].

6. The case of negative integral central charge

For each integer $n \geq 1$, $\mathcal{V}_{-n}$ admits a free field realization as the Sp_{2n}-invariant subalgebras of the $\beta\gamma$-system $\mathcal{S}(V)$ for $V = \mathbb{C}^n$, by Corollary 13.1 of [KWY]. This indicates that the structure of $\mathcal{V}_{-n}$ is deeply connected to the invariant theory of the symplectic group. The $\beta\gamma$-system $\mathcal{S}(V)$, or algebra of chiral differential operators on V, was introduced in [FMS]. It is the unique even vertex algebra with generators β^x, $\gamma^{x'}$ for $x \in V$, $x' \in V^*$, which satisfy the OPE relations

$$\beta^x(z)\gamma^{x'}(w) \sim \langle x', x \rangle (z-w)^{-1}, \qquad \gamma^{x'}(z)\beta^x(w) \sim -\langle x', x \rangle (z-w)^{-1},$$

$$\beta^x(z)\beta^y(w) \sim 0, \qquad \gamma^{x'}(z)\gamma^{y'}(w) \sim 0. \tag{6.1}$$

There is a one-parameter family of conformal structures

$$L_\lambda = \lambda \sum_{i=1}^n :\beta^{x_i} \partial \gamma^{x'_i}: + (\lambda - 1) \sum_{i=1}^n :\partial \beta^{x_i} \gamma^{x'_i}: \tag{6.2}$$

of central charge $n(12\lambda^2 - 12\lambda + 2)$, under which β^x and $\gamma^{x'}$ are primary of conformal weights λ and $1 - \lambda$, respectively. Here $\{x_1, \dots, x_n\}$ is a basis for V and $\{x'_1, \dots, x'_n\}$ is the dual basis for V^*. For $\lambda \neq \frac{1}{2}$, the group of automorphisms of $\mathcal{S}(V)$ preserving L_λ is

GL_n, and the vector spaces $\{\beta^x | x \in V\}$ and $\{\gamma^{x'} | x' \in V^*\}$ transform as copies of V and V^*, respectively. For $\lambda = \frac{1}{2}$, the group of conformal automorphisms is augmented to Sp_{2n}, which is the *full* automorphism group of $\mathcal{S}(V)$, and $\{\beta^x, \gamma^{x'} | x \in V,\ x' \in V^*\}$ is a copy of the standard Sp_{2n}-module $\mathbb{C}^{2n}$. *For the rest of this paper, $\mathcal{S}(V)$ will be equipped with the conformal structure $L_{1/2}$.*

There is a basis of $\mathcal{S}(V)$ consisting of the normally ordered monomials

$$:\partial^{I_1}\beta^{x_1}\cdots\partial^{I_n}\beta^{x_n}\partial^{J_1}\gamma^{x'_1}\cdots\partial^{J_n}\gamma^{x'_n}:, \tag{6.3}$$

In this notation, $I_k = (i^k_1, \dots, i^k_{r_k})$ and $J_k = (j^k_1, \dots, j^k_{s_k})$ are lists of integers satisfying $0 \le i^k_1 \le \cdots \le i^k_{r_k}$ and $0 \le j^k_1 \le \cdots \le j^k_{s_k}$, and

$$\partial^{I_k}\beta^{x_k} = :\partial^{i^k_1}\beta^{x_k}\cdots\partial^{i^k_{r_k}}\beta^{x_k}:, \qquad \partial^{J_k}\gamma^{x'_k} = :\partial^{j^k_1}\gamma^{x'_k}\cdots\partial^{j^k_{s_k}}\gamma^{x'_k}: \ .$$

We have a $\mathbb{Z}_{\geq 0}$-grading

$$\mathcal{S}(V) = \bigoplus_{d\geq 0}\mathcal{S}(V)^{(d)}, \tag{6.4}$$

where $\mathcal{S}(V)^{(d)}$ is spanned by monomials of the form (6.3) of total degree $d = \sum_{k=1}^n r_k + s_k$. Finally, we define the filtration $\mathcal{S}(V)_{(d)} = \bigoplus_{i=0}^d \mathcal{S}(V)^{(i)}$. This filtration satisfies (3.2) and (3.3), and we have

$$\mathrm{gr}(\mathcal{S}(V)) \cong \mathrm{Sym}\bigoplus_{k\geq 0} U_k, \tag{6.5}$$

where U_k is the copy of the standard $2n$-dimensional irreducible Sp_{2n}-module spanned by $\{\beta^x_k, \gamma^{x'}_k |\ x \in V,\ x' \in V^*\}$. In this notation, β^x_k and $\gamma^{x'}_k$ are the images of $\partial^k\beta^x(z)$ and $\partial^k\gamma^{x'}(z)$ in $\mathrm{gr}(\mathcal{S}(V))$ under the projection $\phi_1 : \mathcal{S}(V)_{(1)} \to \mathcal{S}(V)_{(1)}/\mathcal{S}(V)_{(0)}$.

Theorem 6.1. *(Kac-Wang-Yan) There is an isomorphism $\mathcal{V}_{-n} \to \mathcal{S}(V)^{Sp_{2n}}$ given by*

$$w^m \mapsto \frac{1}{2}\sum_{i=1}^n \big(:\beta^{x_i}\partial^m\gamma^{x'_i}: - :\partial^m\beta^{x_i}\gamma^{x'_i}: \big), \qquad m \in 1 + 2\mathbb{Z}_{\geq 0}. \tag{6.6}$$

This map preserves conformal weight and is a morphism in the category $\mathcal{R}$. Since the action of Sp_{2n} on $\mathcal{S}(V)$ preserves the grading (6.4), $\mathcal{V}_{-n}$ is a graded subalgebra of $\mathcal{S}(V)$. We write

$$\mathcal{V}_{-n} = \bigoplus_{d\geq 0}(\mathcal{V}_{-n})^{(d)}, \qquad (\mathcal{V}_{-n})^{(d)} = \mathcal{V}_{-n} \cap \mathcal{S}(V)^{(d)}, \tag{6.7}$$

and define the corresponding filtration by $(\mathcal{V}_{-n})_{(d)} = \bigoplus_{i=0}^d (\mathcal{V}_{-n})^{(i)}$.

The identification $\mathcal{V}_{-n} \cong \mathcal{S}(V)^{Sp_{2n}}$ suggests an alternative strong generating set for $\mathcal{V}_{-n}$ coming from classical invariant theory. Since the action of Sp_{2n} on $\mathcal{S}(V)$ preserves the filtration, we have

$$\mathrm{gr}(\mathcal{V}_{-n}) \cong \mathrm{gr}(\mathcal{S}(V)^{Sp_{2n}}) \cong \mathrm{gr}(\mathcal{S}(V))^{Sp_{2n}} \cong (\mathrm{Sym}\bigoplus_{k\geq 0} U_k)^{Sp_{2n}}. \tag{6.8}$$

The generators and relations for $(\mathrm{Sym}\bigoplus_{k\geq 0} U_k)^{Sp_{2n}}$ are given by Weyl's *first and second fundamental theorems of invariant theory* for the standard representation of Sp_{2n} [We].

Theorem 6.2. *(Weyl) For $k \geq 0$, let U_k be the copy of the standard Sp_{2n}-module $\mathbb{C}^{2n}$ with symplectic basis $\{x_{i,k}, y_{i,k}|\ i = 1, \dots, n\}$. The invariant ring $(Sym \bigoplus_{k\geq 0} U_k)^{Sp_{2n}}$ is generated by the quadratics*

$$q_{a,b} = \frac{1}{2}\sum_{i=1}^{n} \big(x_{i,a}y_{i,b} - x_{i,b}y_{i,a}\big), \qquad 0 \leq a < b. \tag{6.9}$$

For $a > b$, define $q_{a,b} = -q_{b,a}$, and let $\{Q_{a,b}|\ a, b \geq 0\}$ be commuting indeterminates satisfying $Q_{a,b} = -Q_{b,a}$ and no other algebraic relations. The kernel I_n of the homomorphism

$$\mathbb{C}[Q_{a,b}] \to (Sym \bigoplus_{k\geq 0} U_k)^{Sp_{2n}}, \qquad Q_{a,b} \mapsto q_{a,b}, \tag{6.10}$$

is generated by the degree $n+1$ Pfaffians p_I, which are indexed by lists $I = (i_0, \dots, i_{2n+1})$ of integers satisfying

$$0 \leq i_0 < \cdots < i_{2n+1}. \tag{6.11}$$

For $n = 1$ and $I = (i_0, i_1, i_2, i_3)$, we have

$$p_I = q_{i_0,i_1}q_{i_2,i_3} - q_{i_0,i_2}q_{i_1,i_3} + q_{i_0,i_3}q_{i_1,i_2},$$

and for $n > 1$ they are defined inductively by

$$p_I = \sum_{r=1}^{2n+1} (-1)^{r+1} q_{i_0,i_r} p_{I_r}, \tag{6.12}$$

where $I_r = (i_1, \dots, \widehat{i_r}, \dots, i_{2n+1})$ is obtained from I by omitting i_0 and i_r.

Under the identification (6.8), the generators $q_{a,b}$ correspond to strong generators

$$\omega_{a,b} = \frac{1}{2}\sum_{i=1}^{n} \big(\ :\partial^a \beta^{x_i} \partial^b \gamma^{x'_i}: - :\partial^b \beta^{x_i} \partial^a \gamma^{x'_i}:\big), \qquad 0 \leq a < b, \tag{6.13}$$

of $\mathcal{V}_{-n}$, satisfying $\phi_2(\omega_{a,b}) = q_{a,b}$. In this notation, we have

$$w^{2m+1} = \omega_{0,2m+1}, \qquad m \geq 0. \tag{6.14}$$

Note that the $\omega_{a,b}$'s are *not* the same as the fields denoted by $\omega_{a,b}$ in [LI]. For each $m \geq 0$, let A_m denote the vector space with basis $\{\omega_{a,b}|\ a + b = m\}$. For $m \geq 0$, we have

$$\dim(A_{2m}) = m, \qquad \dim(A_{2m+1}) = m + 1.$$

Moreover, $\partial(A_m) \subset A_{m+1}$, and we have

$$\dim\big(A_{2m+1}/\partial(A_{2m})\big) = 1, \qquad \dim\big(A_{2m}/\partial(A_{2m-1})\big) = 0. \tag{6.15}$$

Hence A_1 is spanned by w^1 and for $m > 0$, A_{2m+1} has a decomposition

$$A_{2m+1} = \partial(A_{2m}) \oplus \langle w^{2m+1}\rangle = \partial^2(A_{2m-1}) \oplus \langle w^{2m+1}\rangle, \tag{6.16}$$

where $\langle w^{2m+1}\rangle$ is the linear span of w^{2m+1}. Similarly, A_2 is spanned by ∂w^1 and for $m > 0$, A_{2m+2} has a decomposition

$$A_{2m+2} = \partial^2(A_{2m}) \oplus \langle \partial w^{2m+1}\rangle = \partial^3(A_{2m-1}) \oplus \langle \partial w^{2m+1}\rangle. \tag{6.17}$$

It is easy to see that $\{\partial^{2i}w^{2m+1-2i}|\ 0\leq i\leq m\}$ and $\{\partial^{2i+1}w^{2m+1-2i}|\ 0\leq i\leq m\}$ are bases of A_{2m+1} and A_{2m+2}, respectively. For $a+b=2m+1$ and $c+d=2m+2$, $\omega_{a,b}\in A_{2m+1}$ and $\omega_{c,d}\in A_{2m+2}$ can be expressed uniquely in the form

$$\omega_{a,b}=\sum_{i=0}^{m}\lambda_i\partial^{2i}w^{2m+1-2i},\qquad \omega_{c,d}=\sum_{i=0}^{m}\mu_i\partial^{2i+1}w^{2m+1-2i} \tag{6.18}$$

for constants λ_i,μ_i, $i=0,\dots,m$. Hence $\{\partial^k w^{2m+1}|\ k,m\geq 0\}$ and $\{\omega_{a,b}|\ 0\leq a<b\}$ are related by a linear change of variables, and it will be convenient to pass back and forth between these sets. Using (6.18), which holds in $\mathcal{V}_{-n}$ for all n, we can define an alternative strong generating set $\{\Omega_{a,b}|\ 0\leq a<b\}$ for $\mathcal{M}^+_{-n}$ by the same formula: for $a+b=2m+1$ and $c+d=2m+2$,

$$\Omega_{a,b}=\sum_{i=0}^{m}\lambda_i\partial^{2i}W^{2m+1-2i},\qquad \Omega_{c,d}=\sum_{i=0}^{m}\mu_i\partial^{2i+1}W^{2m+1-2i}.$$

Clearly $\pi_{-n}(\Omega_{a,b})=\omega_{a,b}$. We will use the same notation A_m to denote the linear span of $\{\Omega_{a,b}|\ a+b=m\}$, when no confusion can arise.

7. The structure of the ideal $\mathcal{I}_{-n}$

Recall that the projection $\pi_{-n}:\mathcal{M}^+_{-n}\to\mathcal{V}_{-n}$ with kernel $\mathcal{I}_{-n}$ sending $\Omega_{a,b}\mapsto\omega_{a,b}$ is a morphism in the category $\mathcal{R}$. Under the identifications

$$\mathrm{gr}(\mathcal{M}^+_{-n})\cong\mathbb{C}[Q_{a,b}],\qquad \mathrm{gr}(\mathcal{V}_{-n})\cong(\mathrm{Sym}\bigoplus_{k\geq 0}U_k)^{Sp_{2n}}\cong\mathbb{C}[q_{a,b}]/I_n,$$

$\mathrm{gr}(\pi_{-n})$ is just the quotient map (6.10).

Lemma 7.1. *For each $I=(i_0,i_1,\dots,i_{2n+1})$, there exists a unique vertex operator*

$$P_I\in(\mathcal{M}^+_{-n})_{(2n+2)}\cap\mathcal{I}_{-n} \tag{7.1}$$

of weight $n+1+\sum_{a=0}^{2n+1}i_a$, satisfying

$$\phi_{2n+2}(P_I)=p_I. \tag{7.2}$$

These elements generate $\mathcal{I}_{-n}$ as a vertex algebra ideal.

Proof. Clearly π_{-n} maps each filtered piece $(\mathcal{M}^+_{-n})_{(k)}$ onto $(\mathcal{V}_{-n})_{(k)}$, so the hypotheses of Lemma 3.2 are satisfied. Since $I_n=\mathrm{Ker}(\mathrm{gr}(\pi_{-n}))$ is generated by the Pfaffians p_I, we can apply Lemma 3.2 to find $P_I\in(\mathcal{M}^+_{-n})_{(2n+2)}\cap\mathcal{I}_{-n}$ satisfying $\phi_{2n+2}(P_I)=p_I$, such that $\{P_I\}$ generates $\mathcal{I}_{-n}$. If P'_I also satisfies (7.2), we would have $P_I-P'_I\in(\mathcal{M}^+_{-n})_{(2n)}\cap\mathcal{I}_{-n}$. Since there are no relations in $\mathcal{V}_{-n}$ of degree less than $2n+2$, we have $P_I-P'_I=0$. $\square$

Let $\langle P_I\rangle$ denote the vector space with basis $\{P_I\}$ where I satisfies (6.11). We have $\langle P_I\rangle=(\mathcal{M}^+_{-n})_{(2n+2)}\cap\mathcal{I}_{-n}$, and clearly $\langle P_I\rangle$ is a module over the Lie algebra $\mathcal{P}^+\subset\hat{\mathcal{D}}^+$ generated by $\{W^{2m+1}(k)|\ m,k\geq 0\}$, since $\mathcal{P}^+$ preserves both the filtration on $\mathcal{M}^+_{-n}$ and the ideal $\mathcal{I}_{-n}$. It will be convenient to work in the basis

$$\{\Omega_{a,b}(a+b-w)|\ 0\leq a<b,\ a+b-w\geq 0\}$$

for $\mathcal{P}^+$. Note that $\Omega_{a,b}(a+b-w)$ is homogeneous of weight w. The action of $\mathcal{P}^+$ by derivations of degree zero on $\mathrm{gr}(\mathcal{M}^+_{-n})$ coming from the vertex Poisson algebra structure

is independent of n, and is specified by the action of $\mathcal{P}^+$ on the generators $\Omega_{l,m}$. We compute

$$\Omega_{a,b}(a+b-w)(\Omega_{l,m}) = \lambda_{a,b,w,l}(\Omega_{l+w,m}) + \lambda_{a,b,w,m}(\Omega_{l,m+w}), \tag{7.3}$$

where

$$\lambda_{a,b,w,l} = \begin{cases} (-1)^{b+1}\frac{(b+l)!}{2(l+w-a)!} - (-1)^{a+1}\frac{(a+l)!}{2(l+w-b)!} & l+w-a \geq 0 \\ 0 & l+w-a<0 \end{cases}. \tag{7.4}$$

The action of $\mathcal{P}^+$ on $\langle P_I\rangle$ is by "weighted derivation" in the following sense. Given $I = (i_0,\dots,i_{2n+1})$ and $p = \Omega_{a,b}(a+b-w) \in \mathcal{P}^+$, we have

$$p(P_I) = \sum_{r=0}^{2n+1} \lambda_r P_{I^r}, \tag{7.5}$$

for lists $I^r = (i_0,\dots,i_{r-1}, i_r+w, i_{r+1},\dots,i_{2n+1})$, and constants λ_r. If $i_r + w = i_s$ for some $s = 1,\dots,2n+1$ we have $\lambda_r = 0$, and otherwise $\lambda_r = (-1)^k \lambda_{a,b,w,i_r}$, where k is the number of transpositions required to transform I^r into an increasing list, as in (6.11).

For each $n\geq 1$, there is a distinguished element $P_0 \in \langle P_I\rangle$, defined by

$$P_0 = P_I, \qquad I = (0,1,\dots,2n+1).$$

It is the unique element of $\mathcal{I}_{-n}$ of minimal weight $2(n+1)^2$, and hence is a singular vector in $\mathcal{M}^+_{-n}$.

Theorem 7.2. *P_0 generates $\mathcal{I}_{-n}$ as a vertex algebra ideal.*

We need a preliminary result in order to prove this statement. For simplicity of notation, we take $n=1$, but the result we are going to prove holds for any n. In this case, $\mathcal{S} = \mathcal{S}(V)$ is generated by β,γ. Recall from (6.5) that β_j and γ_j denote the images of $\partial^j\beta$ and $\partial^j\gamma$ in $\mathrm{gr}(\mathcal{S})$, respectively. Let $W \subset \mathrm{gr}(\mathcal{S})$ be the vector space with basis $\{\beta_j,\gamma_j|\ j\geq 0\}$, and for each $m\geq 0$, let W_m be the subspace with basis $\{\beta_j,\gamma_j|\ 0\leq j\leq m\}$. Let $\phi: W\to W$ be a linear map of weight $w\geq 1$, such that

$$\phi(\beta_j) = c_j\beta_{j+w}, \qquad \phi(\gamma) = c_j\gamma_{j+w} \tag{7.6}$$

for constants $c_j\in\mathbb{C}$. For example, the restriction of $w^{2k+1}(2k+1-w)$ to W is such a map for $2k+1-w\geq 0$.

Lemma 7.3. *Fix $w\geq 1$ and $m\geq 0$, and let ϕ be a linear map satisfying (7.6). Then the restriction $\phi\big|_{W_m}$ can be expressed uniquely as a linear combination of the operators $w^{2k+1}(2k+1-w)\big|_{W_m}$ for $0\leq 2k+1-w\leq 2m+1$.*

Proof. Suppose first that w is even, and let $k_j = j+\frac{w}{2}$, for $j = 0,\dots,m$. In this notation, we need to show that $\phi\big|_{W_m}$ can be expressed uniquely as a linear combination of the operators $w^{2k_j+1}(2j+1)\big|_{W_m}$ for $j=0,\dots,m$. We calculate

$$w^{2k_j+1}(2j+1)(\beta_i) = \lambda_{0,2k_j+1,w,i}(\beta_{i+w}), \qquad w^{2k_j+1}(2j+1)(\gamma_i) = \lambda_{0,2k_j+1,w,i}(\gamma_{i+w}), \tag{7.7}$$

where $\lambda_{0,2k_j+1,w,i}$ is given by (7.4). Let M^w be the $(m+1)\times(m+1)$ matrix with entries $M^w_{i,j} = \lambda_{0,2k_j+1,w,i}$, for $i,j = 0,\dots,m$. Let $\mathbf{c}\in\mathbb{C}^{m+1}$ be the column vector whose transpose is $(c_0,\dots,c_m)$. Given an arbitrary linear combination

$$\psi = t_0w^{2k_0+1}(1) + t_1w^{2k_1+1}(3) + \cdots + t_mw^{2k_m+1}(2m+1)$$

of the operators $w^{2k_j+1}(2j+1)$ for $0 \leq j \leq m$, let $\mathbf{t}$ be the column vector whose transpose is $(t_0, \dots, t_m)$. Note that $\phi\big|_{W_m} = \psi\big|_{W_m}$ precisely when $M^w \mathbf{t} = \mathbf{c}$, so in order to prove the claim, it suffices to show that M^w is invertible. But this is clear from the fact that each 2×2 minor

$$\begin{bmatrix} M^w_{i,j} & M^w_{i,j+1} \\ M^w_{i+1,j} & M^w_{i+1,j+1} \end{bmatrix}$$

has positive determinant. Finally, if w is odd, the same argument shows that for $k_j = j + \frac{1}{2}(w-1)$, $j = 0, \dots, m$, ϕ can be expressed uniquely as a linear combination of the operators $w^{2k_j+1}(2j)$. $\square$

Since (7.7) holds for all $n \geq 1$ with β_i and γ_i replaced with β^x_i and $\gamma^{x'}_i$, it follows that the statement of Lemma 7.3 holds for any n. More precisely, let $W \subset \text{gr}(\mathcal{S}(V))$ be the vector space with basis $\{\beta^{x_i}_j, \gamma^{x'_i}_j | i = 1, \dots, n, \ j \geq 0\}$, and let $W_m \subset W$ be the subspace with basis $\{\beta^{x_i}_j, \gamma^{x'_i}_j | i = 1, \dots, n, \ 0 \leq j \leq m\}$. Let $\phi : W \to W$ be a linear map of weight $w \geq 1$ taking

$$\beta^{x_i}_j \mapsto c_j \beta^{x_i}_{j+w}, \qquad \gamma^{x'_i}_j \mapsto c_j \gamma^{x'_i}_{j+w}, \qquad i = 1, \dots, n, \tag{7.8}$$

where c_j is independent of i. Then $\phi\big|_{W_m}$ can be expressed uniquely as a linear combination of $w^{2k+1}(2k+1-w)\big|_{W_m}$ for $0 \leq 2k+1-w \leq 2m+1$.

Proof of Theorem 7.2. Since $\mathcal{I}_{-n}$ is generated by $\langle P_I \rangle$ as a vertex algebra ideal, it suffices to show that $\langle P_I \rangle$ is generated by P_0 as a module over $\mathcal{P}^+$. Let $\mathcal{I}'_{-n}$ denote the ideal in $\mathcal{M}^+_{-n}$ generated by P_0, and let $\langle P_I \rangle[k]$, $\mathcal{I}_{-n}[k]$, and $\mathcal{I}'_{-n}[k]$ denote the homogeneous subspaces of weight k. Note that these spaces are all trivial for $k < 2(n+1)^2$, and all are spanned by P_0 for $k = 2(n+1)^2$.

We proceed by induction on k for $k > 2(n+1)^2$, so assume that $\mathcal{I}_{-n}[m-1] = \mathcal{I}'_{-n}[m-1]$. Fix $I = (i_0, i_1, \dots, i_{2n+1})$ such that P_I has weight $m = n+1+\sum_{k=0}^{2n+1} i_k$. Since $m > 2(n+1)^2$, there is some k for which $i_k > k$. Let k be the first element where this happens, and let $I' = (i_0, \dots, i_{k-1}, i_k - 1, i_{k+1}, \dots, i_{2n+1})$. Since $i_{k-1} = k-1$, we have $i_k - 1 > i_{k-1}$, so $P_{I'} \neq 0$. By Lemma 7.3, we can find $p \in \mathcal{P}^+$ such that

$$p(\beta^{x_i}_r) = c_r \beta^{x_i}_{r+1}, \quad p(\gamma^{x'_i}_r) = c_r \gamma^{x_i}_{r+1}$$

where $c_r = 1$ for $r = i_k - 1$ and $c_r = 0$ for all other $r \leq i_{2n+1}$. It is immediate from the weighted derivation property (7.5) that $p(P_{I'}) = P_I$. Therefore $P_I \in \mathcal{I}'_{-n}$. $\square$

8. Normal ordering and quantum corrections

Given a homogeneous polynomial $p \in \text{gr}(\mathcal{M}^+_{-n}) \cong \mathbb{C}[Q_{a,b}]$ of degree k in the variables $Q_{a,b}$, a *normal ordering* of p will be a choice of normally ordered polynomial $P \in (\mathcal{M}^+_{-n})_{(2k)}$, obtained by replacing $Q_{a,b}$ by $\Omega_{a,b}$, and replacing ordinary products with iterated Wick products of the form (2.2). Of course P is not unique, but for any choice of P we have $\phi_{2k}(P) = p$. For the rest of this section, P^{2k}, E^{2k}, F^{2k}, etc., will denote elements of $(\mathcal{M}^+_{-n})_{(2k)}$ which are homogeneous, normally ordered polynomials of degree k in the vertex operators $\Omega_{a,b}$.

Let $P^{2n+2}_I \in (\mathcal{M}^+_{-n})_{(2n+2)}$ be some normal ordering of p_I, so that $\phi_{2n+2}(P^{2n+2}_I) = p_I$. Then

$$\pi_{-n}(P^{2n+2}_I) \in (\mathcal{V}_{-n})_{(2n)},$$

and $\phi_{2n}(\pi_{-n}(P_I^{2n+2})) \in \mathrm{gr}(\mathcal{V}_{-n})$ can be expressed uniquely as a polynomial of degree n in the variables $q_{a,b}$. Choose some normal ordering of the corresponding polynomial in the variables $\Omega_{a,b}$, and call this vertex operator $-P_I^{2n}$. Then $P_I^{2n+2} + P_I^{2n}$ has the property that

$$\phi_{2n+2}(P_I^{2n+2} + P_I^{2n}) = p_I, \qquad \pi_{-n}(P_I^{2n+2} + P_I^{2n}) \in (\mathcal{V}_{-n})_{(2n-2)}.$$

Continuing this process, we arrive at a vertex operator $\sum_{k=1}^{n+1} P_I^{2k}$ in the kernel of π_{-n}, such that $\phi_{2n+2}(\sum_{k=1}^{n+1} P_I^{2k}) = p_I$. By Lemma 7.1, must have

$$P_I = \sum_{k=1}^{n+1} P_I^{2k}. \tag{8.1}$$

In this decomposition, the term P_I^2 lies in the space A_m spanned by $\{\Omega_{a,b}|\ a+b=m\}$, for $m = n + \sum_{a=0}^{2n+1} i_a$. By (6.16), for all odd integers $m \geq 1$ we have a natural projection

$$\mathrm{pr}_m : A_m \to \langle W^m \rangle.$$

For all $I = (i_0, i_1, \dots, i_{2n+1})$ such that $m = n + \sum_{a=0}^{2n+1} i_a$ is odd, define the *remainder*

$$R_I = \mathrm{pr}_m(P_I^2). \tag{8.2}$$

Lemma 8.1. *Fix $P_I \in \mathcal{I}_{-n}$ with $I = (i_0, i_1, \dots, i_{2n+1})$ and $m = n + \sum_{a=0}^{2n+1} i_a$ odd, as above. Suppose that $P_I = \sum_{k=1}^{n+1} P_I^{2k}$ and $P_I = \sum_{k=1}^{n+1} \tilde{P}_I^{2k}$ are two different decompositions of P_I of the form (8.1). Then*

$$P_I^2 - \tilde{P}_I^2 \in \partial^2(A_{m-2}).$$

In particular, R_I is independent of the choice of decomposition of P_I.

Proof. Since $\mathcal{M}_{-n}^+$ is a vertex subalgebra of $\mathcal{M}_{-n}$ and the generators of $\mathcal{M}_{-n}^+$ are linear combinations of the generators of $\mathcal{M}_{-n}$ and their derivatives, this is an immediate consequence of Lemma 4.7 of [LI]. □

Lemma 8.2. *Let R_0 denote the remainder of the element P_0. The condition $R_0 \neq 0$ is equivalent to the existence of a decoupling relation in $\mathcal{V}_{-n}$ of the form*

$$w^{2n^2+4n+1} = Q(w^1, w^3, w^5, \dots, w^{2n^2+4n-1}), \tag{8.3}$$

where Q is a normally ordered polynomial in $w^1, w^3, w^5, \dots, w^{2n^2+4n-1}$ and their derivatives.

Proof. Let $P_0 = \sum_{k=1}^{n+1} P_0^{2k}$ be a decomposition of P_0 of the form (8.1). If $R_0 \neq 0$, we have $P_0^2 = \lambda W^{2n^2+4n+1} + \partial^2 \omega$ for some $\lambda \neq 0$ and $\omega \in A_{2n^2+4n-1}$. Since P_0 has weight $2(n+1)^2$ and P_0^{2k} has degree k in the variables W^{2m+1} and their derivatives, P_0^{2k} must depend only on $W^1, W^3, W^5, \dots, W^{2n^2+4n-1}$ and their derivatives, for $2 \leq k \leq n+1$. It follows that $\frac{1}{\lambda} P_0$ has the form

$$W^{2n^2+4n+1} - Q(W^1, W^3, W^5, \dots, W^{2n^2+4n-1}). \tag{8.4}$$

Applying the projection $\pi_{-n} : \mathcal{M}_{-n}^+ \to \mathcal{V}_{-n}$ yields the result, since $\pi_{-n}(P_0) = 0$. The converse follows from the fact that P_0 is the unique element of $\mathcal{I}_{-n}$ of weight $2(n+1)^2$, up to scalar multiples. □

Lemma 8.3. *Suppose that $R_0 \neq 0$. Then there exist higher decoupling relations*

$$w^{2m+1} = Q_m(w^1, w^3, w^5, \dots, w^{2n^2+4n-1}) \tag{8.5}$$

for all $m \geq n^2+2n$, where Q_m is a normally ordered polynomial in $w^1, w^3, w^5, \dots, w^{2n^2+4n-1}$, and their derivatives. It follows that $\{w^1, w^3, w^5, \dots, w^{2n^2+4n-1}\}$ is a minimal strong generating set for $\mathcal{V}_{-n}$.

Proof. It suffices to find elements

$$W^{2m+1} - Q_m(W^1, W^3, W^5, \dots, W^{2n^2+4n-1}) \in \mathcal{I}_{-n},$$

so we assume inductively that Q_l exists for $n^2+2n \leq l < m$. Choose a decomposition

$$Q_{m-1} = \sum_{k=1}^{d} Q^{2k}_{m-1},$$

where Q^{2k}_{m-1} is a homogeneous normally ordered polynomial of degree k in the vertex operators $W^1, W^3, W^5, \dots, W^{2n^2+4n-1}$ and their derivatives. In particular,

$$Q^2_{m-1} = \sum_{i=0}^{n^2+2n-1} c_i \partial^{2m-2i-2} W^{2i+1},$$

for constants $c_0, \dots, c_{n^2+2n-1}$. We apply the operator $W^3 \circ_1 \in \mathcal{P}^+$, which raises the weight by two. By (5.2), we have

$$W^3 \circ_1 W^{2m-1} = -2mW^{2m+1} + \sum_{k=1}^{m} \lambda_k \partial^{2k} W^{2m+1-2k},$$

for constants λ_k. Since Q_l exists for $n^2+2n \leq l < m$, whenever $2m+1-2k > 2n^2+4n-1$, we can use the element

$$\partial^{2k}\bigg(W^{2m+1-2k} - Q_{m-k}(W^1, W^3, W^5, \dots, W^{2n^2+4n-1})\bigg) \in \mathcal{I}_{-n}$$

to express $\partial^{2k} W^{2m+1-2k}$ as a normally ordered polynomial in $W^1, W^3, W^5, \dots, W^{2n^2+4n-1}$ and their derivatives, modulo $\mathcal{I}_{-n}$.

Moreover, $W^3 \circ_1 \big(\sum_{k=1}^{d} Q^{2k}_{m-1}\big)$ can be expressed in the form $\sum_{k=1}^{d} E^{2k}$, where each E^{2k} is a normally ordered polynomial in $W^1, W^3, W^5, \dots, W^{2n^2+4n+1}$ and their derivatives. If W^{2n^2+4n+1} or its derivatives appear in E^{2k}, we can use the element (8.4) in $\mathcal{I}_{-n}$ to eliminate W^{2n^2+4n+1} and any of its derivatives, modulo $\mathcal{I}_{-n}$. Hence

$$W^3 \circ_1 \big(\sum_{k=1}^{d} Q^{2k}_{m-1}\big)$$

can be expressed modulo $\mathcal{I}_{-n}$ in the form $\sum_{k=1}^{d'} F^{2k}$, where $d' \geq d$, and F^{2k} is a normally ordered polynomial in $W^1, W^3, W^5, \dots, W^{2n^2+4n-1}$ and their derivatives. It follows that

$$-\frac{1}{2m} W^3 \circ_1 \bigg(W^{2m-1} - Q_{m-1}(W^1, W^3, W^5, \dots, W^{2n^2+4n-1})\bigg)$$

can be expressed as an element of $\mathcal{I}_{-n}$ of the desired form. $\square$

9. A CLOSED FORMULA FOR R_I

We shall find a closed formula for R_I for any $I = (i_0, i_1, \dots, i_{2n+1})$ such that $\mathrm{wt}(P_I) = n + 1 + \sum_{a=0}^{2n+1} i_a$ is even, and it will be clear from our formula that $R_0 \neq 0$. We introduce the notation

$$R_I = R_n(I)W^m, \qquad m = n + \sum_{a=0}^{2n+1} i_a \tag{9.1}$$

so that $R_n(I)$ denotes the coefficient of W^m in $\mathrm{pr}_m(P_I^2)$. For $n = 1$ and $I = (i_0, i_1, i_2, i_3)$ the following formula is easy to obtain using the fact $\mathrm{pr}_m(\Omega_{a,b}) = (-1)^m W^m$ for $m = a + b$.

$$\begin{aligned} R_1(I) = \frac{1}{4}\Bigg(&\frac{(-1)^{i_0+i_2} - (-1)^{2i_0+i_1+i_2} - (-1)^{i_0+i_3} + (-1)^{2i_0+i_1+i_3}}{1 + i_0 + i_1} \\ &+ \frac{-(-1)^{i_0+i_1} + (-1)^{2i_0+i_1+i_2} + (-1)^{i_0+i_3} - (-1)^{2i_0+i_2+i_3}}{1 + i_0 + i_2} \\ &+ \frac{(-1)^{i_0+i_1} - (-1)^{i_0+i_2} - (-1)^{2i_0+i_1+i_3} + (-1)^{2i_0+i_2+i_3}}{1 + i_0 + i_3} \\ &+ \frac{-(-1)^{i_0+2i_1+i_2} + (-1)^{i_0+i_1+2i_2} - (-1)^{i_1+i_3} + (-1)^{i_2+i_3}}{1 + i_1 + i_2} \\ &+ \frac{(-1)^{i_1+i_2} + (-1)^{i_0+2i_1+i_3} - (-1)^{i_2+i_3} - (-1)^{i_0+i_1+2i_3}}{1 + i_1 + i_3} \\ &+ \frac{-(-1)^{i_1+i_2} + (-1)^{i_1+i_3} - (-1)^{i_0+2i_2+i_3} + (-1)^{i_0+i_2+2i_3}}{1 + i_2 + i_3}\Bigg). \end{aligned} \tag{9.2}$$

Before we can find a closed formula for $R_n(I)$ for all n, we will find a *recursive* formula for $R_n(I)$ in terms of the expressions $R_{n-1}(J)$, so we will assume that $R_{n-1}(J)$ has been defined for all J. Recall first that $\mathcal{S}(V)$ is a graded algebra with $\mathbb{Z}_{\geq 0}$ grading (6.4), which specifies a linear isomorphism

$$\mathcal{S}(V) \cong \mathrm{Sym} \bigoplus_{k\geq 0} U_k, \qquad U_k \cong \mathbb{C}^{2n}.$$

Since $\mathcal{V}_{-n}$ is a graded subalgebra of $\mathcal{S}(V)$, we obtain an isomorphism of graded vector spaces

$$i_{-n} : \mathcal{V}_{-n} \to (\mathrm{Sym} \bigoplus_{k\geq 0} U_k)^{Sp_{2n}}. \tag{9.3}$$

Let $p \in (\mathrm{Sym} \bigoplus_{k\geq 0} U_k)^{Sp_{2n}}$ be a homogeneous polynomial of degree $2d$, and let $f = (i_{-n})^{-1}(p) \in (\mathcal{V}_{-n})^{(2d)}$ be the corresponding homogeneous vertex operator. Let $F \in (\mathcal{M}_{-n}^+)_{(2d)}$ be a vertex operator satisfying $\pi_{-n}(F) = f$, where $\pi_{-n} : \mathcal{M}_{-n}^+ \to \mathcal{V}_{-n}$ is the projection. We can write $F = \sum_{k=1}^d F^{2k}$, where F^{2k} is a normally ordered polynomial of degree k in the vertex operators $\Omega_{a,b}$.

Next, let $\tilde{V}$ be the vector space $\mathbb{C}^{n+1}$, and let

$$\tilde{q}_{a,b} \in (\text{Sym} \bigoplus_{k\geq 0} \tilde{U}_k)^{Sp_{2n+2}} \cong \text{gr}(\mathcal{S}(\tilde{V})^{Sp_{2n+2}}) \cong \text{gr}(\mathcal{V}_{-n-1})$$

be the generator given by (6.9), where $\tilde{U}_k \cong \mathbb{C}^{2n+2}$. Let $\tilde{p}$ be the polynomial of degree $2d$ obtained from p by replacing each $q_{a,b}$ with $\tilde{q}_{a,b}$, and let $\tilde{f} = (i_{-n-1})^{-1}(\tilde{p}) \in (\mathcal{V}_{-n-1})^{(2d)}$ be the corresponding homogeneous vertex operator. Finally, let $\tilde{F}^{2k} \in \mathcal{M}^+_{-n-1}$ be the vertex operator obtained from F^{2k} by replacing each $\Omega_{a,b}$ with the corresponding vertex operator $\tilde{\Omega}_{a,b} \in \mathcal{M}^+_{-n-1}$, and let $\tilde{F} = \sum_{i=1}^{d} \tilde{F}^{2k}$.

Lemma 9.1. *Fix $n \geq 1$, and let P_I be an element of $\mathcal{I}_{-n}$ given by Lemma 7.1. There exists a decomposition $P_I = \sum_{k=1}^{n+1} P_I^{2k}$ of the form* (8.1) *such that the corresponding vertex operator*

$$\tilde{P}_I = \sum_{k=1}^{n+1} \tilde{P}_I^{2k} \in \mathcal{M}^+_{-n-1}$$

has the property that $\pi_{-n-1}(\tilde{P}_I)$ lies in the homogeneous subspace $(\mathcal{V}_{-n-1})^{(2n+2)}$ of degree $2n+2$.

Proof. The argument is the same as the proof of Corollary 4.14 of [LI], and is omitted. □

Recall that the Pfaffian p_I has an expansion

$$p_I = \sum_{r=1}^{2n+1} (-1)^{r+1} q_{i_0,i_r} p_{I_r},$$

where $I_r = (i_1, \dots, \widehat{i_r}, \dots, i_{2n+1})$ is obtained from I by omitting i_0 and i_r. Let $P_{I_r} \in \mathcal{M}^+_{-n+1}$ be the vertex operator corresponding to p_{I_r}. By Lemma 9.1, there exists a decomposition

$$P_{I_r} = \sum_{i=1}^{n} P_{I_r}^{2i}$$

such that the corresponding element $\tilde{P}_{I_r} = \sum_{i=1}^{n} \tilde{P}_{I_r}^{2i} \in \mathcal{M}^+_{-n}$ has the property that $\pi_{-n}(\tilde{P}_{I_r})$ lies in the homogeneous subspace $(\mathcal{V}_{-n})^{(2n)}$ of degree $2n$. We have

$$\sum_{r=1}^{2n+1} (-1)^{r+1} :\Omega_{i_0,i_r} \tilde{P}_{I_r}: = \sum_{r=1}^{2n+1} \sum_{i=1}^{n} (-1)^{r+1} :\Omega_{i_0,i_r} \tilde{P}_{I_r}^{2i}: . \tag{9.4}$$

The right hand side of (9.4) consists of normally ordered monomials of degree at least 2 in the vertex operators $\Omega_{a,b}$, and hence contributes nothing to $R_n(I)$. Since $\pi_{-n}(\tilde{P}_{I_r})$ is homogeneous of degree $2n$, $\pi_{-n}(:\Omega_{i_0,i_r} \tilde{P}_{I_r}:)$ consists of a piece of degree $2n+2$ and a piece of degree $2n$ coming from all double contractions of Ω_{i_0,i_r} with terms in $\tilde{P}_{I_r}$, which lower the degree by two. The component of

$$\pi_{-n}\bigg(\sum_{r=1}^{2n+1} (-1)^{r+1} :\Omega_{i_0,i_r} \tilde{P}_{I_r}: \bigg) \in \mathcal{V}_{-n}$$

in degree $2n+2$ must cancel since this sum corresponds to the classical Pfaffian p_I, which is a relation among the variables $q_{a,b}$. The component of $:\Omega_{i_0,i_r}\tilde{P}_{I_r}:$ in degree $2n$ is

$$S_r = \frac{1}{2}\Bigg((-1)^{i_0}\sum_a \frac{\tilde{P}_{I_{r,a}}}{i_0+i_a+1} + (-1)^{i_r+1}\sum_a \frac{\tilde{P}_{I_{r,a}}}{i_r+i_a+1}\Bigg) \tag{9.5}$$

In this notation, for $a \in \{i_0,\dots,i_{2n+1}\}\backslash\{i_0,i_r\}$, $I_{r,a}$ is obtained from $I_r = (i_1,\dots,\widehat{i_r},\dots,i_{2n+1})$ by replacing i_a with $i_a+i_0+i_r+1$. It follows that

$$\pi_{-n}\Bigg(\sum_{r=1}^{2n+1}(-1)^{r+1}:\Omega_{i_0,i_r}\tilde{P}_{I_r}:\Bigg) = \pi_{-n}\Bigg(\sum_{r=1}^{2n+1}(-1)^{r+1}S_r\Bigg). \tag{9.6}$$

Combining (9.4) and (9.6), we can regard

$$\sum_{r=1}^{2n+1}\sum_{i=1}^{n}(-1)^{r+1}:\Omega_{i_0,i_r}\tilde{P}_{I_r}^{2i}: - \sum_{r=0}^{n}(-1)^{r+1}S_r$$

as a decomposition of P_I of the form $P_I = \sum_{k=1}^{n+1}P_I^{2k}$ where the leading term $P_I^{2n+2} = \sum_{r=0}^{2n+1}(-1)^{r+1}:\Omega_{i_0,i_r}\tilde{P}_{I_r}^{2n}:$. It follows that $R_n(I)$ is the negative of the sum of the terms $R_{n-1}(J)$ corresponding to each $\tilde{P}_J$ appearing in $\sum_{r=0}^{2n+1}(-1)^{r+1}S_r$. We obtain the following recursive formula:

$$R_n(I) = -\frac{1}{2}\sum_{r=1}^{2n+1}(-1)^{r+1}\Bigg((-1)^{i_0}\sum_a \frac{R_{n-1}(I_{r,a})}{i_0+i_a+1} + (-1)^{i_r+1}\sum_a \frac{R_{n-1}(I_{r,a})}{i_r+i_a+1}\Bigg). \tag{9.7}$$

To find a closed formula for $R_n(I)$, we begin with the case $n=1$. It follows from (9.2) that $R_1(I)=0$ unless $I=(i_0,i_1,i_2,i_3)$ contains two even and two odd elements. By permuting i_0,i_1,i_2,i_3 if necessary, we may assume that $i_k \equiv k \bmod 2$. In this case, (9.2) simplifies as follows:

$$R_1(I) = \frac{(2+i_0+i_1+i_2+i_3)(i_0-i_2)(i_1-i_3)}{(1+i_0+i_1)(1+i_1+i_2)(1+i_0+i_3)(1+i_2+i_3)}. \tag{9.8}$$

It is clear by induction on n that $R_n(I)=0$ unless $I=(i_0,\dots,i_{2n+1})$ is "balanced" in the sense that it has $n+1$ even elements and $n+1$ odd elements. From now on, we assume this is the case, and we change our notation slightly. We write $I=(i_0,j_0,i_1,j_1,\dots,i_n,j_n)$, where $i_0,\dots,i_n$ are even and $j_0,\dots,j_n$ are odd. For later use, we record one more obvious symmetry: for $I=(i_0,j_0,i_1,j_1,\dots,i_n,j_n)$ with $i_0,\dots,i_n$ even and $j_0,\dots j_n$ odd, we have

$$R_n(I) = (-1)^{n+1}R_n(I'), \qquad I' = (j_0,i_0,j_1,i_1,\dots,j_n,i_n). \tag{9.9}$$

Fix $I=(i_0,j_0,i_1,j_1,\dots,i_{n-1},j_{n-1})$ with $i_0,i_1,\dots,i_{n-1}$ even and $j_0,j_1,\dots,j_{n-1}$ odd. For each even integer $x\geq 0$, set $I_x=(i_0,j_0,i_1,j_1,\dots,i_{n-1},j_{n-1},x,x+1)$. We regard $R_n(I_x)$ as a rational function of x. By applying the recursive formula (9.7) k times, we can express $R_n(I_x)$ as a linear combination of terms of the form $R_m(K)$ where $m=n-k$ and $K=(k_0,k_1,\dots,k_{2m+1})$. Each entry of K is a constant plus a linear combination of entries from I_x, and at most two entries of K depend on x. Let V_m denote the vector space spanned by elements $R_m(K)$ with these properties. Using (9.7), we can express $R_m(K)$ as a linear

combination of elements in V_{m-1}. A term $R_{m-1}(K_{r,a})$ in this decomposition will be called *x-active* if either k_0, k_r, or k_a depends on x. Define linear maps

$$f_m : V_m \to V_{m-1}, \qquad g_m : V_m \to V_{m-1} \tag{9.10}$$

as follows: $f_m(R_m(K))$ is the sum of the terms which are not x-active, and $g_m(R_m(K))$ is the sum of the terms which are x-active. Finally, define the *constant term map*

$$h_m : V_m \to \mathbb{Q}, \qquad h_m(R_m(K)) = \lim_{x\to\infty} R_m(K). \tag{9.11}$$

Lemma 9.2. *For all $n \geq 2$, and all $I = (i_0, j_0, i_1, j_1, \dots, i_{n-1}, j_{n-1})$ with $i_0, \dots, i_{n-1}$ even and $j_0, \dots, j_{n-1}$ odd, we have*

$$h_n(R_n(I_x)) = \frac{n}{n + \sum_{k=0}^{n-1} i_k + j_k} R_{n-1}(I). \tag{9.12}$$

Proof. For $n = 2$ this is an easy calculation, so we may proceed by induction on n. In fact, it will be convenient to prove the following auxiliary formula at the same time:

$$h_{n-1}(g_n(R_n(I_x))) = \frac{1}{n + \sum_{k=0}^{n-1} i_k + j_k} R_{n-1}(I). \tag{9.13}$$

For $n = 2$ this can be checked by direct calculation, so we assume both (9.12) and (9.13) for $n-1$. Each term appearing in $f_n(R_n(I_x))$ is of the form

$$R_{n-1}(K), \qquad K = (k_0, k_1, \dots, k_{2n-3}, x, x+1), \qquad n - 1 + \sum_{t=0}^{2n-3} k_t = n + \sum_{k=0}^{n-1} i_k + j_k.$$

By our inductive hypothesis that (9.12) holds for $n-1$, we have

$$h_{n-1}(f_n(R_n(I_x))) = \frac{n-1}{n + \sum_{k=0}^{n-1} i_k + j_k} R_{n-1}(I). \tag{9.14}$$

Next, it is easy to check that

$$h_{n-2}(g_{n-1}(f_n(R_n(I_x)))) = h_{n-2}(f_{n-1}(g_n(R_n(I_x)))).$$

Also, we have

$$g_{n-1}(g_n(R_n(I_x))) = 0,$$

since all terms in this expression cancel pairwise. Therefore

$$\begin{aligned} h_{n-1}(g_n(R_n(I_x))) &= h_{n-2}(f_{n-1}(g_n(R_n(I_x)))) + h_{n-2}(g_{n-1}(g_n(R_n(I_x)))) \\ &= h_{n-2}(f_{n-1}(g_n(R_n(I_x)))) = h_{n-2}(g_{n-1}(f_n(R_n(I_x)))). \end{aligned}$$

Moreover, by applying the induction hypothesis that (9.13) holds for $n-1$, it follows that

$$h_{n-1}(g_n(R_n(I_x))) = h_{n-2}(g_{n-1}(f_n(R_n(I_x)))) = \frac{1}{n + \sum_{k=0}^{n-1} i_k + j_k} R_{n-1}(I). \tag{9.15}$$

Since $h_n(R_n(I_x) = h_{n-1}(f_n(R_n(I_x))) + h_{n-1}(g_n(R_n(I_x)))$, the claim follows from (9.14) and (9.15). ☐

Theorem 9.3. *Suppose that $I = (i_0, j_0, i_1, j_1, \dots, i_n, j_n)$ is a list of nonnegative integers such that $i_0, \dots, i_n$ are even and $j_0, \dots, j_n$ are odd, as above. Then the following closed formula holds:*

$$R_n(I) = \frac{n!\Big(n + 1 + \sum_{k=0}^{n} i_k + j_k\Big)\Big(\prod_{0\leq k<l\leq n}(i_k - i_l)(j_k - j_l)\Big)}{\prod_{0\leq k\leq n,\ 0\leq l\leq n}(1 + i_k + j_l)}. \tag{9.16}$$

Proof. As we shall see, this formula is more or less forced upon us by the symmetries of the Pfaffians, which are inherited by the numbers $R_n(I)$. We assume that (9.16) holds for $n-1$ and all $K=(k_0,l_0,k_1,l_1,\dots,k_{n-1},l_{n-1})$ with k_i even and l_i odd, and we proceed by induction on n. In particular, each term of the form $R_{n-1}(K)$ is a rational function in the entries of K, where the denominator has degree n^2 and the numerator has degree n^2-n+1. Hence the total degree of $R_{n-1}(K)$ is $-n+1$.

Using (9.7), we may expand $R_n(I)$ as a rational function of $i_0,j_0,i_1,j_1,\dots,i_n,j_n$. Each term appearing in (9.7) has degree $-n$, so $R_n(I)$ has degree at most $-n$. The denominator of each such term is clearly a product of factors of the form

$$1+i_k+j_l,\quad 1+i_k+i_l,\quad 1+j_k+j_l,\quad 2+i_0+i_k+i_l,\quad 2+i_0+j_k+i_l,\quad 2+i_0+j_k+j_l.$$

As a rational function of $i_0,j_0,i_1,j_1,\dots,i_n,j_n$, the denominator of $R_n(I)$ is therefore the product of a subset of these terms. Moreover, $R_n(I)$ has the following symmetry; for every interchange of i_k and i_l, we pick up a sign, and for every interchange of j_k and j_l we pick up a sign. These permutations must have the effect of permuting the factors in the denominator up to a sign, and permuting the factors of the numerator up to a sign.

From this symmetry, it is clear that none of the expressions

$$2+i_0+i_k+i_l,\qquad 2+i_0+j_k+i_l,\qquad 2+i_0+j_k+j_l$$

can appear. Moreover, each term $R_{n-1}(I_{r,a})$ appearing in (9.7) with i_a and i_r both even must vanish, since $I_{r,a}$ will then contain $n-2$ even elements and $n+2$ odd elements. Hence none of the factors $1+i_k+i_l$ can appear in the denominator of $R_n(I)$. By (9.9), the factors $1+j_k+j_l$ also cannot appear, so the denominator of $R_n(I)$ can contain only a subset of expressions of the form $1+i_k+j_l$. By symmetry, it must contain *all* such expressions, so the denominator is precisely

$$\prod_{0\le k\le n,\ 0\le l\le n}(1+i_k+j_l),$$

and therefore has leading degree $(n+1)^2$. Since $R_n(I)$ picks up a sign under permutation of i_k and i_l, and under permutation of j_k and j_l, the numerator must be divisible by

$$\prod_{0\le k<l\le n}(i_k-i_l)(j_k-j_l),$$

which is homogeneous of degree n^2+n. Since the denominator has degree $(n+1)^2$, the total degree of $R_n(I)$ is at least $-n-1$.

We have already seen that the total degree of $R_n(I)$ is at most $-n$, so there is room for at most one more linear factor in the numerator. By symmetry, this factor must be invariant under all permutations of $i_0,\dots,i_n$ and all permutations of $j_0,\dots,j_n$, so it is of the form

$$a_n\left(\sum_{k=0}^n i_k\right)+b_n\left(\sum_{k=0}^n j_k\right)+c_n,$$

where a_n,b_n,c_n are constants which depend on n but not on I. The additional symmetry (9.9) shows that $a_n=b_n$, so this can be rewritten in the form

$$a_n\left(\sum_{k=0}^n i_k+j_k\right)+c_n.$$

In order to complete the proof of Theorem 9.3, it suffices to show that

$$a_n = n!, \qquad c_n = (n+1)!. \tag{9.17}$$

By our inductive assumption, $a_{n-1} = (n-1)!$. Fix $K = (k_0, l_0, k_1, l_1, \dots, k_{n-1}, l_{n-1})$ with k_r even and l_r odd. As in Lemma 9.2, for each even integer $x \geq 0$, set

$$K_x = (k_0, l_0, k_1, l_1, \dots, k_{n-1}, l_{n-1}, x, x+1).$$

The highest power of x appearing in the numerator of $R_n(K_x)$ is x^{2n+1}, and the coefficient of x^{2n+1} in the numerator is

$$2a_n \prod_{0 \leq r < s < n} (k_r - k_s)(l_r - l_s).$$

Similarly, the highest power of x appearing in the denominator of $R_n(K_x)$ is also x^{2n+1}, and the coefficient of x^{2n+1} is

$$2 \prod_{0 \leq r < n,\ 0 \leq s < n} (1 + k_r + l_s).$$

Therefore

$$h_n(R_n(K_x)) = \lim_{x \to \infty} R_n(K_x) = \frac{a_n \prod_{0 \leq r < s < n} (k_r - k_s)(l_r - l_s)}{\prod_{0 \leq r < n,\ 0 \leq s < n} (1 + k_r + l_s)}.$$

By our inductive assumption, we have

$$R_{n-1}(K) = \frac{(n-1)!(n + \sum_{r=0}^{n-1} k_r + l_r) \prod_{0 \leq r < s < n} (k_r - k_s)(l_r - l_s)}{\prod_{0 \leq r < n,\ 0 \leq s < n} (1 + k_r + l_s)}.$$

Therefore by (9.12), we have

$$h_n(R_n(K_x)) = \left(\frac{n}{n + \sum_{r=0}^{n-1} k_r + l_r} \right) \frac{(n-1)!(n + \sum_{r=0}^{n-1} k_r + l_r) \prod_{0 \leq r < s < n} (k_r - k_s)(l_r - l_s)}{\prod_{0 \leq r < n,\ 0 \leq s < n} (1 + k_r + l_s)}.$$

This proves that $a_n = n!$.

Finally, we need to show that $c_n = (n+1)!$. Let $I = (i_0, j_0, i_1, j_1, \dots, i_n, j_n)$ as above. Since $a_n = n!$, it suffices to show that the numerator of $R_n(I)$ is divisible by $n + 1 + \sum_{k=0}^{n} i_k + j_k$. But this is clear from (9.7), since by inductive assumption, the numerator of each term of the form $R_{n-1}(I_{r,a})$ is divisible by $n + 1 + \sum_{k=0}^{n} i_k + j_k$. □

Theorem 9.4. *For all $n \geq 1$, $\mathcal{V}_{-n}$ has a minimal strong generating set $\{w^1, w^3, w^5, \dots, w^{2n^2+4n-1}\}$, and is therefore a $\mathcal{W}$-algebra of type $\mathcal{W}(2, 4, 6, \dots, 2n^2 + 4n)$.*

Proof. Specializing (9.16) to the case $I = (0, 1, \dots, 2n+1)$ yields

$$R_n(I) = \frac{2(n!)(n+1)^2 \prod_{0 \leq k < l \leq n} (k - l)^2}{\prod_{0 \leq k \leq n,\ 0 \leq l \leq n} (1 + k + l)}.$$

In particular, $R_0 = R_n(I) W^{2n^2+4n+1} \neq 0$, so the claim follows from Lemma 8.3. □

The basic tool in studying the representation theory of vertex algebras is the *Zhu functor*, which was introduced by Zhu in [Z]. Given a vertex algebra $\mathcal{W}$ with weight grading $\mathcal{W} = \bigoplus_{n\in\mathbb{Z}} \mathcal{W}_n$, this functor attaches to $\mathcal{W}$ an associative algebra $A(\mathcal{W})$, together with a surjective linear map $\pi_{\text{Zhu}} : \mathcal{W} \to A(\mathcal{W})$. For $a \in \mathcal{W}_m$ and $b \in \mathcal{W}$, define

$$a * b = \text{Res}_z \left(a(z) \frac{(z+1)^m}{z} b \right), \tag{10.1}$$

and extend $*$ by linearity to a bilinear operation $\mathcal{W} \otimes \mathcal{W} \to \mathcal{W}$. Let $O(\mathcal{W})$ denote the subspace of $\mathcal{W}$ spanned by elements of the form

$$a \circ b = \text{Res}_z \left(a(z) \frac{(z+1)^m}{z^2} b \right) \tag{10.2}$$

where $a \in \mathcal{W}_m$, and let $A(\mathcal{W})$ be the quotient $\mathcal{W}/O(\mathcal{W})$, with projection $\pi_{\text{Zhu}} : \mathcal{W} \to A(\mathcal{W})$. Then $O(\mathcal{W})$ is a two-sided ideal in $\mathcal{W}$ under the product $*$, and $(A(\mathcal{W}), *)$ is a unital, associative algebra. The assignment $\mathcal{W} \mapsto A(\mathcal{W})$ is functorial, and if $\mathcal{I}$ is a vertex algebra ideal of $\mathcal{W}$, we have $A(\mathcal{W}/\mathcal{I}) \cong A(\mathcal{W})/I$, where $I = \pi_{\text{Zhu}}(\mathcal{I})$. A well-known formula asserts that for all $a \in \mathcal{W}_m$ and $b \in \mathcal{W}$,

$$a * b - b * a \equiv \text{Res}_z (1+z)^{m-1} a(z) b \mod O(\mathcal{W}). \tag{10.3}$$

A $\mathbb{Z}_{\geq 0}$-graded module $M = \bigoplus_{n\geq 0} M_n$ over $\mathcal{W}$ is called *admissible* if for every $a \in \mathcal{W}_m$, $a(n) M_k \subset M_{m+k-n-1}$, for all $n \in \mathbb{Z}$. Given $a \in \mathcal{W}_m$, $a(m-1)$ acts on each M_k. The subspace M_0 is then a module over $A(\mathcal{W})$ with action $[a] \mapsto a(m-1) \in \text{End}(M_0)$. In fact, $M \mapsto M_0$ provides a one-to-one correspondence between irreducible, admissible $\mathcal{W}$-modules and irreducible $A(\mathcal{W})$-modules. If $A(\mathcal{W})$ is a commutative algebra, all its irreducible modules are one-dimensional, and the corresponding $\mathcal{W}$-modules $M = \bigoplus_{n\geq 0} M_n$ are cyclic and generated by any nonzero $v \in M_0$. Accordingly, we call such a module a *highest-weight module* for $\mathcal{W}$, and we call v a *highest-weight vector*.

Let $\mathcal{W}$ be a vertex algebra which is strongly generated by elements α_i of weights w_i, for i in some index set I. Then $A(\mathcal{W})$ is generated by $\{a_i = \pi_{\text{Zhu}}(\alpha_i(z)) | \ i \in I\}$. Moreover, $A(\mathcal{W})$ inherits a filtration (but not a grading) by weight.

Theorem 10.1. *For all $n \geq 1$, $A(\mathcal{V}_{-n})$ is a commutative algebra, so all irreducible, admissible $\mathcal{V}_{-n}$-modules are highest-weight modules.*

Proof. It suffices to show that $[a_i, a_j] = 0$ for all $i, j = 1, 3, 5, \dots, 2n^2 + 4n - 1$. This is clear from the fact $A(\mathcal{W}_{1+\infty,-n})$ is abelian, and the generators w^{2m+1} of $\mathcal{V}_{-n}$ are linear combinations of the generators of $\mathcal{W}_{1+\infty,-n}$ and their derivatives. $\square$

The same argument shows that $A(\mathcal{M}^+_{-n})$ is abelian. Since $\mathcal{M}^+_{-n}$ is freely generated by $W^1, W^3, W^5, \dots$ it follows that $A(\mathcal{M}^+_{-n})$ is the polynomial algebra $\mathbb{C}[A^1, A^3, A^5, \cdots]$, where $A^{2m+1} = \pi_{\text{Zhu}}(W^{2m+1})$. Moreover, $A(\mathcal{V}_{-n}) \cong \mathbb{C}[a^1, a^3, a^5, \dots]/I_{-n}$, where $I_{-n} = \pi_{\text{Zhu}}(\mathcal{I}_{-n})$, and we have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}^+_{-n} & \stackrel{\pi_{-n}}{\to} & \mathcal{V}_{-n} \\ \downarrow \pi_{\text{Zhu}} & & \downarrow \pi_{\text{Zhu}} \\ A(\mathcal{M}^+_{-n}) & \stackrel{A(\pi_{-n})}{\to} & A(\mathcal{V}_{-n}) \end{array}. \tag{10.4}$$

Since $A(\mathcal{V}_{-n})$ is generated by $\{a^1, a^3, a^5, \dots, a^{2n^2+4n-1}\}$,

$$A(\mathcal{V}_{-n}) \cong \mathbb{C}[a^1, a^3, a^5, \dots, a^{2n^2+4n-1}]/I_{-n},$$

where I_{-n} is now regarded as an ideal inside $\mathbb{C}[a^1, a^3, a^5, \dots, a^{2n^2+4n-1}]$. The corresponding variety $V(I_{-n}) \subset \mathbb{C}^{n^2+2n}$ then parametrizes the irreducible, admissible modules over $\mathcal{V}_{-n}$. The problem of classifying these modules is equivalent to describing I_{-n}.

11. Invariant subalgebras of $\mathcal{S}(V)$ under arbitrary reductive groups

In this section, we study $\mathcal{S}(V)^G$ for a general reductive group $G \subset Sp_{2n}$. By Theorem 13.2 of [KWY], $\mathcal{S}(V)$ has a decomposition of the form

$$\mathcal{S}(V) \cong \bigoplus_{\nu \in H} L(\nu) \otimes M^{\nu}, \tag{11.1}$$

where H indexes the irreducible, finite-dimensional representations $L(\nu)$ of Sp_{2n}, and the M^{ν}'s are inequivalent, irreducible, highest-weight $\mathcal{V}_{-n}$-modules. The modules M^{ν} appearing in (11.1) have an integrality property; the eigenvalues of $\{w^{2m+1}(2m+1) |\, m \geq 0\}$ on the highest-weight vectors f_{ν} are all integers. These modules therefore correspond to certain rational points on the variety $V(I_{-n})$. Using a classical theorem of Weyl, we show that $\mathcal{S}(V)^G$ has an (infinite) strong generating set that lives in the direct sum of finitely many of the modules M^{ν}. An easy consequence is that $\mathcal{S}(V)^G$ is finitely generated as a vertex algebra. The *strong* finite generation of $\mathcal{S}(V)^G$ takes more work, and our proof is essentially constructive. The key step is to show that the modules M^{ν} appearing in (11.1) have a certain finiteness property. Together with the finite generation of $\mathcal{S}(V)^G$ and the fact that $\mathcal{V}_{-n}$ is of type $\mathcal{W}(2, 4, 6, \dots, 2n^2+4n)$, this is enough to prove the result.

Theorem 11.1. *For any reductive group G of automorphisms of $\mathcal{S}(V)$, $\mathcal{S}(V)^G$ is finitely generated as a vertex algebra.*

Proof. Recall that $\mathcal{S}(V) \cong \text{gr}(\mathcal{S}(V))$ as linear spaces, and

$$\text{gr}(\mathcal{S}(V)^G) \cong (\text{gr}(\mathcal{S}(V))^G \cong (\text{Sym} \bigoplus_{k \geq 0} U_k)^G = R$$

as commutative algebras, where $U_k \cong \mathbb{C}^{2n}$ as Sp_{2n}-modules. For all $p \geq 0$, there is an action of GL_p on $\bigoplus_{k=0}^{p-1} U_k$ which commutes with the action of G. The natural inclusions $GL_p \hookrightarrow GL_q$ for $p < q$ sending

$$M \rightarrow \begin{bmatrix} M & 0 \\ 0 & I_{q-p} \end{bmatrix}$$

induce an action of $GL_{\infty} = \lim_{p \to \infty} GL_p$ on $\bigoplus_{k \geq 0} U_k$. We obtain an action of GL_{∞} on $\text{Sym} \bigoplus_{k \geq 0} U_k$ by algebra automorphisms, which commutes with the action of G. Hence GL_{∞} acts on R as well. By a basic theorem of Weyl, R is generated by the set of translates under GL_{∞} of any set of generators for $(\text{Sym} \bigoplus_{k=0}^{2n-1} U_k)^G$ [We]. Since G is reductive, $(\text{Sym} \bigoplus_{k=0}^{2n-1} U_k)^G$ is finitely generated, so it contains homogeneous elements $f_1, \dots, f_k$ such that $\{\sigma f_i |\, i = 1, \dots, k,\ \sigma \in GL_{\infty}\}$ generates R. By Lemma 3.1, the set

$$\{(\sigma f_i)(z) \in \mathcal{S}(V)^G |\, i = 1, \dots, k,\ \sigma \in GL_{\infty}\},$$

where $(\sigma f_i)(z)$ correspond to σf_i under the linear isomorphism $\mathcal{S}(V)^G \cong \text{gr}(\mathcal{S}(V)^G) \cong R$, is a strong generating set for $\mathcal{S}(V)^G$.

In the decomposition (11.1) of $\mathcal{S}(V)$ as a bimodule over Sp_{2n} and $\mathcal{V}_{-n}$, the Sp_{2n}-isotypic component of $\mathcal{S}(V)$ of type $L(\nu)$ is isomorphic to $L(\nu)\otimes M^{\nu}$. Each $L(\nu)$ is a module over $G\subset Sp_{2n}$, and since G is reductive, it has a decomposition $L(\nu)=\oplus_{\mu\in H^{\nu}}L(\nu)_{\mu}$. Here μ runs over a finite set H^{ν} of irreducible, finite-dimensional representations $L(\nu)_{\mu}$ of G, possibly with multiplicity. We thus obtain a refinement of (11.1):

$$\mathcal{S}(V)\cong\bigoplus_{\nu\in H}\bigoplus_{\mu\in H^{\nu}}L(\nu)_{\mu}\otimes M^{\nu}. \tag{11.2}$$

Let $f_1(z),\dots,f_k(z)\in\mathcal{S}(V)^G$ correspond to $f_1,\dots,f_k$ under the linear isomorphism $\mathcal{S}(V)^G\cong\mathrm{gr}(\mathcal{S}(V)^G)\cong R$. Clearly $f_1(z),\dots,f_k(z)$ must live in a finite direct sum

$$\bigoplus_{j=1}^{r}L(\nu_j)\otimes M^{\nu_j} \tag{11.3}$$

of the modules appearing in (11.1). By enlarging the collection $f_1(z),\dots,f_k(z)$ if necessary, we may assume without loss of generality that each $f_i(z)$ lives in a single representation of the form $L(\nu_j)\otimes M^{\nu_j}$. Moreover, we may assume that $f_i(z)$ lives in a trivial G-submodule $L(\nu_j)_{\mu_0}\otimes M^{\nu_j}$, where μ_0 denotes the trivial, one-dimensional G-module. (In particular, $L(\nu_j)_{\mu_0}$ is one-dimensional). Since the actions of GL_{∞} and Sp_{2n} on $\mathcal{S}(V)$ commute, we may assume that $(\sigma f_i)(z)\in L(\nu_j)_{\mu_0}\otimes M^{\nu_j}$ for all $\sigma\in GL_{\infty}$. Since $\mathcal{S}(V)^G$ is strongly generated by the set $\{(\sigma f_i)(z)|\ i=1,\dots,k,\ \sigma\in GL_{\infty}\}$, and each M^{ν_j} is an irreducible $\mathcal{V}_{-n}$-module, $\mathcal{S}(V)^G$ is generated as an algebra over $\mathcal{V}_{-n}$ by $f_1(z),\dots,f_k(z)$. Finally, since $\mathcal{V}_{-n}$ is generated by w^3 as a vertex algebra, we conclude that $\mathcal{S}(V)^G$ is finitely generated. $\square$

Next, we need a fact about representations of associative algebras which can be found in [LII]. Let A be an associative $\mathbb{C}$-algebra (not necessarily unital), and let W be a linear representation of A, via an algebra homomorphism $\rho: A\to\mathrm{End}(W)$. Regarding A as a Lie algebra with commutator as bracket, let $\rho_{Lie}: A\to\mathrm{End}(W)$ denote the map ρ, regarded now as a Lie algebra homomorphism. There is an induced algebra homomorphism $U(A)\to\mathrm{End}(W)$, where $U(A)$ denotes the universal enveloping algebra of A. Given elements $a,b\in A$, we denote the product in $U(A)$ by $a*b$ to distinguish it from $ab\in A$. Given a monomial $\mu=a_1*\cdots*a_r\in U(A)$, let $\tilde{\mu}=a_1\cdots a_r$ be the corresponding element of A. Let $U(A)_+$ denote the augmentation ideal (i. e., the ideal generated by A), regarded as an associative algebra with no unit. The map $U(A)_+\to A$ sending $\mu\mapsto\tilde{\mu}$ is then an algebra homomorphism which makes the diagram

$$\begin{array}{ccc} U(A)_+ & & \\ \downarrow & \searrow & \\ A & \to & \mathrm{End}(W) \end{array} \tag{11.4}$$

commute. Let $\mathrm{Sym}(W)$ denote the symmetric algebra of W, whose dth graded component is denoted by $\mathrm{Sym}^d(W)$. Clearly ρ_{Lie} (but not ρ) can be extended to a Lie algebra homomorphism $\hat{\rho}_{Lie}: A\to\mathrm{End}(\mathrm{Sym}(W))$, where $\hat{\rho}_{Lie}(a)$ acts by derivation on each $\mathrm{Sym}^d(W)$:

$$\hat{\rho}_{Lie}(a)(w_1\cdots w_d)=\sum_{i=1}^{d}w_1\cdots\hat{\rho}_{Lie}(a)(w_i)\cdots w_d.$$

This extends to an algebra homomorphism $U(A)\to\mathrm{End}(\mathrm{Sym}(W))$ which we also denote by $\hat{\rho}_{Lie}$, but there is no commutative diagram like (11.4) because the map $A\to\mathrm{End}(\mathrm{Sym}(W))$ is not a map of associative algebras. In particular, the restrictions of $\hat{\rho}_{Lie}(\mu)$

and $\hat{\rho}_{Lie}(\tilde{\mu})$ to $\text{Sym}^d(W)$ are generally not the same for $d > 1$. The following result appears as Lemma 3 of [LII].

Lemma 11.2. *Given $\mu \in U(A)$ and $d \geq 1$, define a linear map $\Phi^d_\mu \in End(Sym^d(W))$ by*

$$\Phi^d_\mu = \hat{\rho}_{Lie}(\mu)\big|_{Sym^d(W)}. \tag{11.5}$$

Let E denote the subspace of $End(Sym^d(W))$ spanned by $\{\Phi^d_\mu |\ \mu \in U(A)\}$. Note that E has a filtration

$$E_1 \subset E_2 \subset \cdots, \qquad E = \bigcup_{r\geq 1} E_r,$$

where E_r is spanned by $\{\Phi^d_\mu |\ \mu \in U(A),\ deg(\mu) \leq r\}$. Then $E = E_d$.

Corollary 11.3. *Let $f \in Sym^d(W)$, and let $M \subset Sym^d(W)$ be the cyclic $U(A)$-module generated by f. Then $\{\hat{\rho}_{Lie}(\mu)(f) |\ \mu \in U(A),\ deg(\mu) \leq d\}$ spans M.*

Recall the Lie algebra $\mathcal{P}^+ \subset \hat{\mathcal{D}}^+$ generated by the modes $\{w^{2m+1}(k)|\ m \geq 0,\ k \geq 0\}$. Note that $\mathcal{P}^+$ has a decomposition

$$\mathcal{P}^+ = \mathcal{P}^+_{<0} \oplus \mathcal{P}^+_0 \oplus \mathcal{P}^+_{>0},$$

where $\mathcal{P}^+_{<0}$, $\mathcal{P}^+_0$, and $\mathcal{P}^+_{>0}$ are the Lie algebras spanned by $\{w^{2m+1}(k)|\ 0 \leq k < 2m+1\}$, $\{w^{2m+1}(2m+1)\}$, and $\{w^{2m+1}(k)|\ k > 2m+1\}$, respectively. Clearly $\mathcal{P}^+$ preserves the filtration on $\mathcal{V}_{-n}$, so each element of $\mathcal{P}^+$ acts by a derivation of degree zero on $\text{gr}(\mathcal{V}_{-n})$.

Let $\mathcal{M}$ be an irreducible, highest-weight $\mathcal{V}_{-n}$-submodule of $\mathcal{S}(V)$ with generator $f(z)$, and let $\mathcal{M}'$ denote the $\mathcal{P}^+$-submodule of $\mathcal{M}$ generated by $f(z)$. Since $f(z)$ has minimal weight among elements of $\mathcal{M}$ and $\mathcal{P}^+_{>0}$ lowers weight, $f(z)$ is annihilated by $\mathcal{P}^+_{>0}$. Moreover, $\mathcal{P}^+_0$ acts diagonalizably on $f(z)$, so $f(z)$ generates a one-dimensional $\mathcal{P}^+_0 \oplus \mathcal{P}^+_{>0}$-module. By the Poincare-Birkhoff-Witt theorem, $\mathcal{M}'$ is a quotient of

$$U(\mathcal{P}^+) \otimes_{U(\mathcal{P}^+_0 \oplus \mathcal{P}^+_{>0})} \mathbb{C}f(z),$$

and in particular is a cyclic $\mathcal{P}^+_{<0}$-module with generator $f(z)$. Suppose that $f(z)$ has degree d, that is, $f(z) \in \mathcal{S}(V)_{(d)} \setminus \mathcal{S}(V)_{(d-1)}$. Since $\mathcal{P}^+$ preserves the filtration on $\mathcal{S}(V)$, and $\mathcal{M}$ is irreducible, the nonzero elements of $\mathcal{M}'$ lie in $\mathcal{S}(V)_{(d)} \setminus \mathcal{S}(V)_{(d-1)}$. Therefore, the projection $\mathcal{S}(V)_{(d)} \to \mathcal{S}(V)_{(d)}/\mathcal{S}(V)_{(d-1)} \subset \text{gr}(\mathcal{S}(V))$ restricts to an isomorphism of $\mathcal{P}^+$-modules

$$\mathcal{M}' \cong \text{gr}(\mathcal{M}') \subset \text{gr}(\mathcal{S}(V)). \tag{11.6}$$

By Corollary 11.3, we conclude that $\mathcal{M}'$ is spanned by elements of the form

$$\{w^{2l_1+1}(k_1)\cdots w^{2l_r+1}(k_r)f(z)|\ w^{2l_i+1}(k_i) \in \mathcal{P}^+_{<0},\ r \leq d\}.$$

The next result is analogous to Lemma 7 of [LII], and the proof is almost identical.

Lemma 11.4. *Let $\mathcal{M}$ be an irreducible, highest-weight $\mathcal{V}_{-n}$-submodule of $\mathcal{S}(V)$ with highest-weight vector $f(z)$ of degree d. Let $\mathcal{M}'$ be the corresponding $\mathcal{P}^+$-module generated by $f(z)$, and let f be the image of $f(z)$ in $gr(\mathcal{S}(V))$, which generates $M = gr(\mathcal{M}')$ as a $\mathcal{P}^+$-module. Fix m so that $f \in Sym^d(W_m)$. Then $\mathcal{M}'$ is spanned by*

$$\{w^{2l_1+1}(k_1)\cdots w^{2l_r+1}(k_r)f(z)|\ w^{2l_i+1}(k_i) \in \mathcal{P}^+_{<0},\ \ r \leq d,\ \ 0 \leq k_i \leq 2m+1\}.$$

As in [LII], we may order the elements $w^{2l+1}(k) \in \mathcal{P}^+_{<0}$ as follows: $w^{2l_1+1}(k_1) > w^{2l_2+1}(k_2)$ if $l_1 > l_2$, or $l_1 = l_2$ and $k_1 < k_2$. Then Lemma 11.4 can be strengthened as follows: $\mathcal{M}'$ is spanned by elements of the form $w^{2l_1+1}(k_1)\cdots w^{2l_r+1}(k_r)f(z)$ with

$$(11.7) \qquad w^{2l_i+1}(k_i) \in \mathcal{P}^+_{<0}, \quad r \leq d, \quad 0 \leq k_i \leq 2m+1, \quad w^{2l_1+1}(k_1) \geq \cdots \geq w^{2l_r+1}(k_r).$$

We use the notation $\mathcal{V}_{-n}[k]$, $\mathcal{M}[k]$, and $\mathcal{M}'[k]$ to denote the homogeneous components of these spaces of conformal weight k. Define the *Wick ideal* $\mathcal{M}_{Wick} \subset \mathcal{M}$ to be the subspace spanned by elements of the form

$$: a(z)b(z) :, \qquad a(z) \in \bigoplus_{k>0} \mathcal{V}_{-n}[k], \qquad b(z) \in \mathcal{M}.$$

Despite the choice of terminology, $\mathcal{M}_{Wick}$ is not a vertex algebra ideal.

Lemma 11.5. *Let $\mathcal{M}$ be an irreducible, highest-weight $\mathcal{V}_{-n}$-submodule of $\mathcal{S}(V)$ with highest-weight vector $f(z)$. Then any homogeneous element of $\mathcal{M}$ of sufficiently high weight lies in the Wick ideal. In particular, $\mathcal{M}/\mathcal{M}_{Wick}$ is finite-dimensional.*

Proof. It suffices to show that $\mathcal{M}'[k]$ lies in the Wick ideal for $k >> 0$, where $\mathcal{M}'$ is the $\mathcal{P}^+$-module generated by $f(z)$. As usual, let d be the degree of $f(z)$, and fix m so that $f \in \text{Sym}^d(W_m)$. Recall that $\mathcal{M}'$ is spanned by elements of the form $w^{2l_1+1}(k_1)\cdots w^{2l_r+1}(k_r)f(z)$ satisfying (11.7). Fix an element $\alpha(z)$ of this form of weight $K >> 0$. Since each operator $w^{2l_i+1}(k_i)$ has weight $2l_i + 1 - k_i$, $k_i \leq 2m+1$, and $K >> 0$, we may assume that $l_1 >> n^2 + 2n$. Then the decoupling relation (8.3) allows us to express $w^{2l_1+1}(z)$ as a normally ordered polynomial $Q_{l_1}(z)$ in the generators

$$(11.8) \qquad \partial^t w^{2l+1}(z), \quad 0 \leq l \leq n^2 + 2n - 1, \quad t \geq 0.$$

We claim that for any weight-homogeneous, normally ordered polynomial $Q(z)$ in the generators (11.8) of sufficiently high weight, any element $c(z) \in \mathcal{M}$, and any k satisfying $0 \leq k \leq 2m+1$, $Q(z) \circ_k c(z)$ lies in $\mathcal{M}_{Wick}$. Specializing this to the case $Q(z) = Q_{l_1}(z)$, $c(z) = w^{2l_2+1}(k_2)\cdots w^{2l_r+1}(k_r)f(z)$, and $k = k_1$, proves the lemma.

We may assume without loss of generality that $Q(z) =: a(z)b(z) :$ where $a(z) = \partial^t w^{2l+1}(z)$ for some $0 \leq l \leq n^2 + 2n - 1$. Suppressing the formal variable z, we have

$$(11.9) \qquad Q \circ_k c = \big(: ab : \big) \circ_k c = \sum_{r \geq 0} \frac{1}{r!} : (\partial^r a)(b \circ_{k+r} c) : + \sum_{r \geq 0} b \circ_{k-r-1} (a \circ_r c),$$

by a well known vertex algebra identity. Suppose first that $b = \lambda 1$ for some constant λ. Then $Q = \lambda \partial^t w^{2l+1}$, and since $\text{wt}(Q) >> 0$, we have $t >> 0$. Hence $Q \circ_k = \lambda(\partial^t w^{2l+1}) \circ_k = 0$ as an operator (since this operator vanishes whenever $t > k$). So we may assume without loss of generality that b is not a constant.

We proceed by induction on k. For $k = 0$, each term appearing in (11.9) lies in $\mathcal{M}_{Wick}$, so there is nothing to prove. For $k > 0$, the only terms appearing in (11.9) that need not lie in $\mathcal{M}_{Wick}$ a priori, are those of the form $\sum_{r=0}^{k-1} b \circ_{k-r-1} (a \circ_r c)$. However, each of these terms is weight-homogeneous, and the weight of $a \circ_r c = \partial^t w^{2l+1} \circ_r c$ is bounded above by $\text{wt}(c) + 2n^2 + 4n - 1$, since $\partial^t w^{2l+1} \circ_r c = 0$ for $t > r$. So we may still assume that $\text{wt}(b) >> 0$. By our inductive assumption, all these terms then lie in $\mathcal{M}_{Wick}$. ∎

Corollary 11.6. *Let $\mathcal{M}$ be an irreducible, highest-weight $\mathcal{V}_{-n}$-submodule of $\mathcal{S}(V)$. Given a subset $S \subset \mathcal{M}$, let $\mathcal{M}_S \subset \mathcal{M}$ denote the subspace spanned by elements of the form*

$$: \omega_1(z) \cdots \omega_t(z)\alpha(z) :, \qquad \omega_j(z) \in \mathcal{V}_{-n}, \qquad \alpha(z) \in S.$$

There exists a finite set $S \subset \mathcal{M}$ such that $\mathcal{M} = \mathcal{M}_S$.

Theorem 11.7. *For any reductive group G of automorphisms of $\mathcal{S}(V)$, $\mathcal{S}(V)^G$ is strongly finitely generated.*

Proof. By Theorem 11.1, we can find vertex operators $f_1(z), \dots, f_k(z)$ such that the corresponding polynomials $f_1, \dots, f_k \in \text{gr}(\mathcal{S}(V))^G$, together with all GL_∞ translates of $f_1, \dots, f_k$, generate the invariant ring $\text{gr}(\mathcal{S}(V))^G$. As in the proof of Lemma 11.1, we may assume that each $f_i(z)$ lies in an irreducible, highest-weight $\mathcal{V}_{-n}$-module $\mathcal{M}_i$ of the form $L(\nu)_{\mu_0} \otimes M^\nu$, where $L(\nu)_{\mu_0}$ is a trivial, one-dimensional G-module. Furthermore, we may assume without loss of generality that $f_1(z), \dots, f_k(z)$ are highest-weight vectors for the action of $\mathcal{V}_{-n}$. Otherwise, we can replace these with the highest-weight vectors in the corresponding modules. For each $\mathcal{M}_i$, choose a finite set $S_i \subset \mathcal{M}_i$ such that $\mathcal{M}_i = (\mathcal{M}_i)_{S_i}$, using Corollary 11.6. Define

$$S = \{w^1, w^3, w^5, \dots, w^{2n^2+4n-1}\} \cup \big(\bigcup_{i=1}^k S_i\big).$$

Since $\{w^1, w^3, w^5, \dots, w^{2n^2+4n-1}\}$ strongly generates $\mathcal{V}_{-n}$, and the set $\bigcup_{i=1}^k \mathcal{M}_i$ strongly generates $\mathcal{S}(V)^G$, it is immediate that S is a strong, finite generating set for $\mathcal{S}(V)^G$. $\square$

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DEPARTMENT OF MATHEMATICS, BRANDEIS UNIVERSITY

E-mail address: `linshaw@brandeis.edu`